

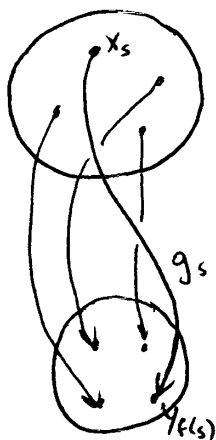
## Composition of Structure Types

Recall that if  $F$  is a structure type &  $Z_0$  is a groupoid,  $F(Z_0)$  is a groupoid whose objects are:

finite sets  $S$ , equipped with an  $F$ -structure,  
with each elt labelled by an object of  $Z_0$ .

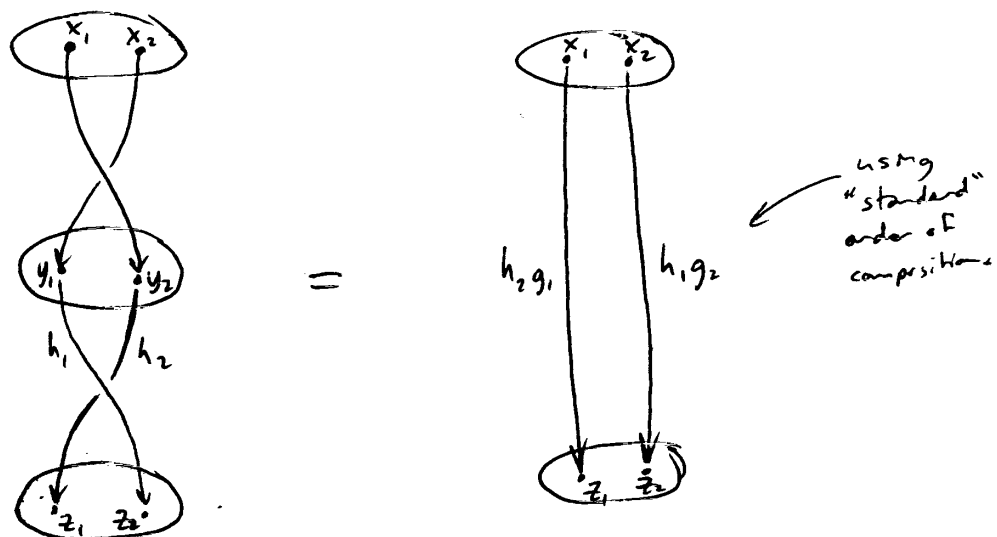
Similarly, a morphism in  $F(Z_0)$  is:

a bijection  $f: S \rightarrow S'$  mapping the  $F$ -str.  
on  $S$  to that on  $S'$ , with each "strand"  
labelled by a morphism  $g_s: x_s \rightarrow y_{f(s)}$  in  $Z_0$ :



where  $x_s$  is the object labelling  $s \in S$   
&  $y_{f(s)}$  is the object labelling  $f(s) \in S'$ .

We compose these morphisms in  $F(Z_0)$  in the obvious way:



Now, given 2 structure types  $F$  &  $G$ , let's try to find a new structure type  $F \circ G$  s.t.

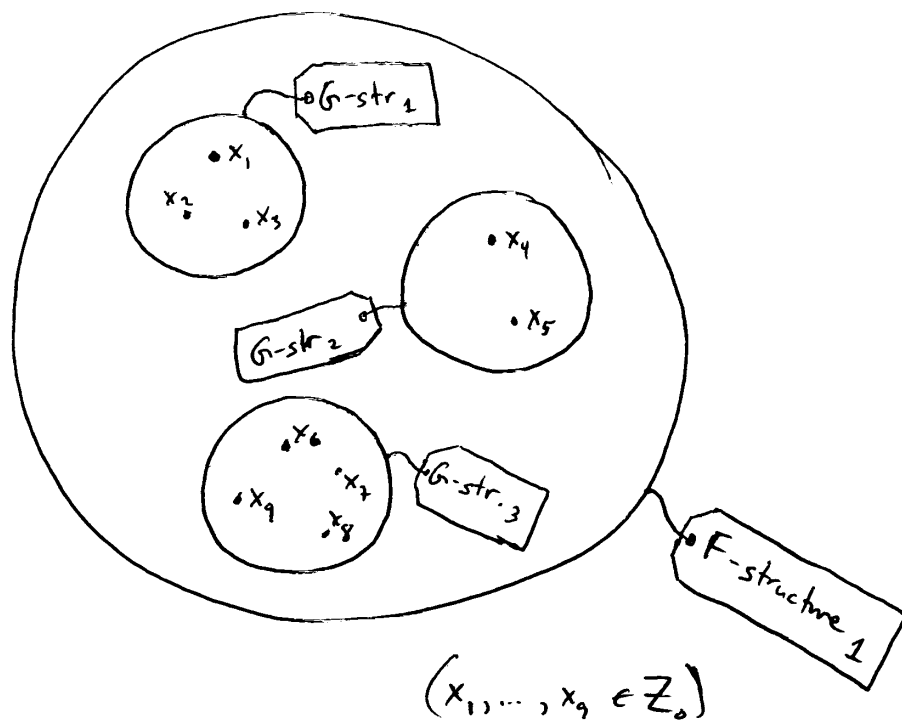
$$(F \circ G)(Z_0) \simeq F(G(Z_0)).$$

$F \circ G$  may not exist (as a structure type — we'll later generalize to stuff types), but when it does, we can figure out what it has to be like, as follows:

$F(G(Z_0)) =$  the groupoid of  $F$ -structured finite sets labelled by  $G(Z_0)$ -objects

$=$  the groupoid of  $F$ -structured finite sets labelled by  $G$ -structured finite sets whose elements are labelled by  $Z_0$  objects.

An object in here looks like



We want a structure type  $F \circ G$  such that objects of  $(F \circ G)(Z_0)$  look like this, so that  $(F \circ G)(Z_0) \cong F(G(Z_0))$ .

$(F \circ G)(Z_0) =$  the groupoid of  $F \circ G$ -structured finite sets labelled by  $Z_0$ -objects.

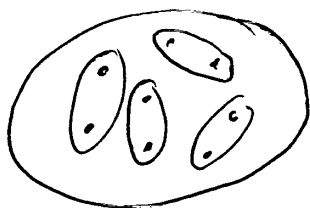
Pondering the picture, we see that to put an  $F \circ G$ -str. on a finite set, we chop it into parts, put a  $G$ -str. on each part & put an  $F$ -str. on the set of parts. (In the picture, we are doing this to a 9 element set).

Example 1

$$G(Z) = \frac{Z^2}{2!} = \text{being a 2-elt. set}$$

$$F(Z) = E^Z = \text{being a finite set}$$

$$F \circ G(Z) \approx E^{Z^2/2!} = \text{"being a finite set chopped up into 2-element sets"}$$



Let's check this:

$$|E^{Z^2/2!}| = \sum_{n \in \mathbb{N}} \frac{a_n Z^n}{n!}$$

where  $a_n = \#$  of ways to chop up an  $n$ -elt set into pairs.

$$\begin{aligned} |E^{Z^2/2!}| &= e^{Z^2/2!} \\ &= \sum_{n \in \mathbb{N}} \frac{(Z^2/2!)^n}{2^n n!} \\ &= \sum_{n \in \mathbb{N}} \frac{Z^{2n}}{2^n n!} \\ &= \sum_{n \in \mathbb{N}} \frac{(2n)!}{2^n n!} \frac{Z^{2n}}{(2n)!} \end{aligned}$$

$$\text{Note: } \frac{(2n)!}{2^n n!} = \frac{1 \cdot 2 \cdot 3 \cdots 2n}{2 \cdot 4 \cdots 2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1) = (2n-1)!!$$

Check:  $n=2, 4$

$$n=2 \quad | \quad 1=1!!$$

$$n=4 \quad | \quad | \quad \times \quad \text{---} \quad 3=3!!$$

We get  $(2n-1) \cdot (2n-3) \cdots 1$  choices since the first element can be mated with any of the  $(2n-1)$  others, the next unmated one has  $(2n-3)$  choices, etc...

(These pictures are simple examples of Feynman diagrams for 2 particles to become 2 particles (if  $n=4$ ))

Example 2: the return of categorified hyperbolic trig:

$$G(Z) = 2Z = \text{"being a 2-colored 1-element set"}$$

$$F(Z) = \text{SINH } Z = \text{"being an odd set"}$$

So

$$(F \circ G)(Z) = \text{SINH } 2Z = \text{"being a 2-colored odd set"}$$

Note:

$$\begin{aligned} \sinh 2z &= \frac{e^{2z} - e^{-2z}}{2} \\ &= \frac{(e^z + e^{-z})(e^z - e^{-z})}{2} \\ &= 2 \cosh z \sinh z \end{aligned}$$

So we can ask: does this come from decategorifying:

$$\begin{aligned} \text{SINH } 2Z &\cong 2 \text{COSH } Z \text{ SINH } Z \quad ? \\ &\cong \text{COSH } Z \text{ SINH } Z + \text{SINH } Z \text{COSH } Z \end{aligned}$$

↑  
"being a 2-colored  
odd set"

↑  
"being a set chopped in 2 parts  
with either first part even & second odd  
or first part odd & second even"

Similarly, how about

$$\begin{aligned} \cosh 2z &= \frac{e^{2z} + e^{-2z}}{2} = \frac{e^{2z} + 2 + e^{-2z}}{4} + \frac{e^{2z} - 2 + e^{-2z}}{4} \\ &= \cosh^2 z + \sinh^2 z \quad ? \end{aligned}$$

Does this come from

$$\cosh z \cong \cosh^2 z + \sinh^2 z$$

"being a 2-colored  
even set"

"being a set  
chopped into two  
even parts"

or

"being a set  
chopped into  
two odd parts"



11 March 2004

If  $F$  is a structure type

$$F(Z) = \sum_{n \in \mathbb{N}} \frac{F_n Z^n}{n!}$$

then we can evaluate it at any groupoid  $Z_0$  and  
get a groupoid

$$F(Z_0) = \sum_{n \in \mathbb{N}} \frac{F_n \times Z_0^n}{n!}$$

&  $F(Z_0)$  is the groupoid of "F-structured finite  
sets with elts labelled by objects of  $Z_0$ ."

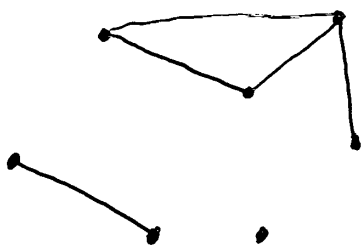
Sometimes, given structure types  $F, G$ , we  
can find a structure type  $F \circ G$  s.t.

$$(F \circ G)(Z_0) \cong F(G(Z_0))$$

Sometimes not; but we can always find a "stuff type"  $F \circ G$  that does the job.

### Example 3 - Logarithms & "connected" structures

Let  $G$  be the structure type of graphs, where for now a graph structure on a finite set  $S$  is like:

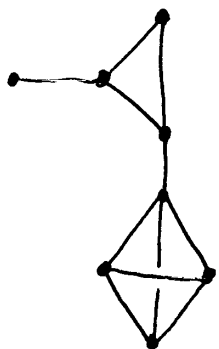


(Rules:

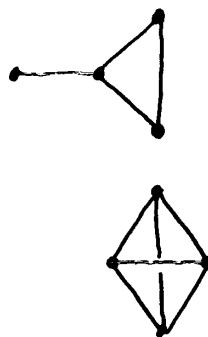
- ① no edges from a vertex to itself
- ② no more than one edge between two vertices

Note: ① & ②  $\Leftrightarrow$  adjacency matrix is any symmetric matrix of 0's & 1's with 0's down the diagonal)

The data just say, for each unordered pair of distinct elts  $x, y \in S$ , whether or not there's an edge between them. Let  $G_{\text{conn}}$  be the structure type of connected graphs:



connected



not connected



There are  $2^{\binom{n}{2}}$  graphs with  $n$  vertices;

$$|G|(z) = \sum_{n \in \mathbb{N}} 2^{\binom{n}{2}} \frac{z^n}{n!}$$

In fact:

$$G(z) = \sum_{n \in \mathbb{N}} 2^{\binom{n}{2}} \frac{z^n}{n!}$$

where

$$\binom{n}{k} = \{ \text{k-elt subsets of } n \}$$

and  $X^Y = \{ f: Y \rightarrow X \}$

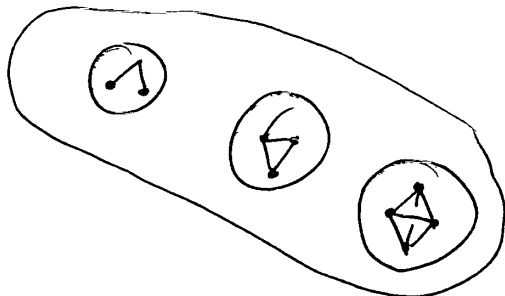
so

$$2^{\binom{n}{2}} = \{ \text{choices of whether or not there's an edge between different elts. of } n \}$$

Now: every graph is a disjoint union of connected ones, so

$$G \cong E^{G_{\text{conn}}}$$

since  $E^{G_{\text{conn}}}(z) =$  "being a finite set partitioned into finite sets equipped with a  $G_{\text{conn}}$  structure"



So:

$$\begin{aligned}
 |G| &= |E^{G_{\text{conn}}}| \\
 &= e^{|G_{\text{conn}}|}
 \end{aligned}$$

so

$$\begin{aligned}
 |G_{\text{conn}}| &= \ln |G| \\
 &= \ln \left( \sum_{n \in \mathbb{N}} 2^{\binom{n}{2}} \frac{z^n}{n!} \right) \\
 &= z + \frac{z^2}{2!} + 4 \frac{z^3}{3!} + 38 \frac{z^4}{4!} + \dots
 \end{aligned}$$

Check:



Yes, there are 4.

Note:

n = 0      The empty graph is not connected!!  
 (It doesn't have exactly one connected component.)

"I think that traditional combinatorists often don't realize that the empty graph is not connected, but they sort of make up for it by not realizing that there is an empty graph."

-J. Dolan

Example 4: Is there a structure type  $F$  such that

$$F(Z_0) \cong E^{E^{Z_0}}$$

for every groupoid  $Z_0$ ? We know that

$E^{E^{Z_0}}$  = "the groupoid of finite sets with elements labelled by finite sets with elements labelled by  $Z_0$ -objects."

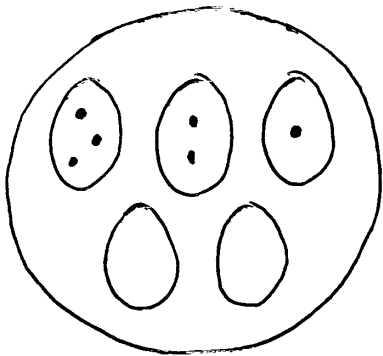
So, if  $Z_0 = 1$ , then

$E^E$  = "the groupoid of finite sets with elements labelled by finite sets."

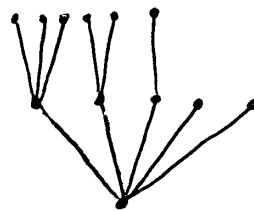
$\simeq$  "finite set bundles over finite sets"

$\simeq$  "rooted trees of height at most 2"  $\leftarrow$  J. Dolan

Typical Example



This is a bird's eye view of this tree:



There's no structure type  $F$  such that

$$E^E = F(1) := \text{"the groupoid of } F\text{-structured sets"} \quad (\text{colored by elts of } 1)$$

If there were such a structure type, then the previous diagram would have to be an example of an  $F$ -structure on a 6-element set. This can't be true because there's more symmetry than we can get from structure — there's more stuff (the empty sets in the diagram)

Suppose there were such an  $F$ :

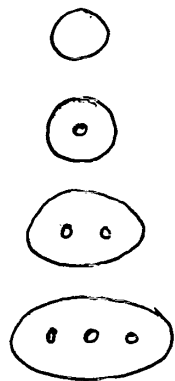
$$|F|(z) = \sum \frac{a_n z^n}{n!}$$

Then  $|F|(0) = a_0 = \#$  of ways to put an  $F$ -str. on the empty set — a cardinal number

But if we take  $E^{E^{Z_0}}$  and set  $Z_0 = 0$  and take the cardinality of the resulting groupoid we get

$$|E^{E^0}| = e^1 = e.$$

In fact  $E^{E^0}$  is the groupoid whose objects look like:



height at most one trees

i.e.  $\text{FinSet}_0$  has cardinality  $e$ .

The Moral: Structure types aren't enough; we need "stuff types" that give not a "set of F-structures on the n-elt set" but a "groupoid of F-stuffs on the n-elt set."

There's a hierarchy:

Examples:

property

"being an even set" is a property of a finite set

structure

"total ordering" is a structure that can be put on a finite set

stuff

"being the first of an ordered pair of sets" is extra stuff added to a finite set.

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Some forgetful functors are more forgetful than others:

(EvenSets)<sub>0</sub>

↓

FinSet<sub>0</sub>

forgets a property

There's also a functor:

(Totally Ordered Finite Sets)<sub>0</sub>



FinSet<sub>0</sub>

forgets a structure

and

$(\text{FinSet}_0)^2 = (\text{FinSet}^2)_0 =$  (ordered pairs of finite sets)



FinSet<sub>0</sub>

forgets stuff  
(throws out a set!)

We'll see that property types  
structure types  
& stuff types

(each of which is a special case of the one after it)  
give groupoids with functors to FinSet<sub>0</sub> that forget

properties  
structure  
& stuff, respectively.

We can generalize everything we've done from  
structure types to stuff types ... and then do  
Feynman diagrams.

Next quarter we'll talk about not only stuff types but stuff operators, which turn one stuff type into another in a linear way. We'll see that the operators in QFT are decategorified versions of stuff operators!

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Two ways to think about structure types:

① Our original definition

$$F: \text{FinSet}_0 \longrightarrow \text{Set}$$

For each finite set, gives you the set of  $F$ -structures on that set. Closely related to taking coefficients of power series.

② Evaluation at a groupoid

$$F: \mathbb{Z}_0 \longmapsto F(\mathbb{Z}_0)$$

This is related to evaluation of power series

① & ② are dual to each other in a way that is essentially like Fourier duality. In fact, here we are categorifying something slightly more mundane than the Fourier transform - something people don't usually talk about. Namely, we are seeing a categorified version of the Taylor transform, which is related to power series in the same way that the Fourier transform is related to Fourier series.

