The Riemann ζ function

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1, 2, 3. *Bernoulli numbers and hyperbolic functions.*

Since

$$\frac{1}{e^z - 1} = \frac{e^{-z/2}}{e^{z/2} - e^{-z/2}},$$

we have

$$\frac{1}{e^z - 1} + \frac{1}{2} = \frac{1}{2}\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}},$$

so

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\coth(z/2).$$

Since cosh is an even function and sinh is an odd function, their ratio coth is odd, and $(z/2)\coth(z/2)$ is an even function of z. This means its Taylor expansion has vanishing odd coefficients, so by the definition of the Bernoulli numbers,

$$B_1 = -\frac{1}{2} \qquad \text{and} \quad B_{2n+1} = 0 \qquad \text{for all } n > 0.$$

Now,

$$z\coth z = \frac{2z}{e^{2z} - 1} + z = \sum_{n\geq 0} \frac{B_{2n}(2z)^{2n}}{(2n)!},$$

so

$$z\cot z = z\,\frac{1}{i}\coth(iz) = \frac{1}{i}\left(\frac{2iz}{e^{2iz} - 1} + iz\right) = \sum_{n\geq 0} \frac{B_{2n}(2iz)^{2n}}{(2n)!}.$$

We can use the fact that $z\cot z$ has poles at $z = \pm\pi$ to estimate of the growth of the Bernoulli numbers. The root test for absolute convergence (or Hadamard's theorem) implies that

$$\lim_{n\to\infty} \sqrt[2n]{\frac{|B_{2n}|(2\pi)^{2n}}{(2n)!}} = 1.$$

With a little undue optimism, we might hope this implies

$$|B_{2n}| \sim \frac{(2n)!}{(2\pi)^{2n}}.$$

By Stirling's approximation, $n! \sim n^n e^{-n}\sqrt{2\pi n}$, we'd then obtain

$$|B_{2n}| \sim \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{(2\pi)^{2n}} = 2\left(\frac{n}{e\pi}\right)^{2n}\sqrt{\pi n}.$$

Unfortunately, the correct asymptotics are

$$|B_{2n}| \sim 4\left(\frac{n}{e\pi}\right)^{2n}\sqrt{\pi n}.$$

Close, but no cigar! The problem is that

$$\lim_{n\to\infty} \sqrt[2n]{\frac{a_n}{b_n}} = 1 \not\Rightarrow a_n \sim b_n.$$

To get the exact asymptotics we'd need to use the 'souped-up Hadamard's theorem' discussed in Week 1 of this quarter. In fact, we'd need to use a slightly enhanced version of this theorem, since there are two different poles closest to the origin, at $\pm\pi$. Perhaps the presence of these two poles with residue 1 is responsible for the mysterious factor of 2?

4. *Enter the Riemann ζ function.*

With the proviso that $\sum_{n\geq 0}$ works only for absolutely convergent sums and that $\sum_{n=0}^{\infty}$ means the limit of partial sums as the upper limit goes to infinity, the given formula is written as

$$\pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\Big[\frac{1}{z-n} + \frac{1}{z+n}\Big] = 1 + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2},$$

which is now absolutely convergent. Then,

$$\pi\cot z = \frac{\pi}{z} + \sum_{n\geq 1}\frac{2\pi z}{z^2-\pi^2n^2},$$

and

$$z\cot z = 1 - 2\sum_{n\geq 1}\frac{z^2}{\pi^2n^2-z^2}.$$

Now,

$$z\cot z = 1 - 2\frac{z^2}{\pi^2}\sum_{n\geq 1}\frac{1}{n^2}\frac{1}{1-(z/n\pi)^2} = 1 - 2\frac{z^2}{\pi^2}\sum_{n\geq 1}\frac{1}{n^2}\sum_{k\geq 0}\Big(\frac{z}{n\pi}\Big)^{2k} = 1 - 2\sum_{k\geq 1}\Big(\frac{z}{\pi}\Big)^{2k}\sum_{n\geq 1}\frac{1}{n^{2k}},$$

so

$$z\cot z = 1 - 2\sum_{k\geq 1}\zeta(2k)\Big(\frac{z}{\pi}\Big)^{2k}.$$

5, 6. *The Riemann ζ function and Bernoulli numbers.*

We now know that

$$\sum_{n\geq 0}\frac{B_{2n}(2iz)^{2n}}{(2n)!} = 1 - 2\sum_{n\geq 1}\zeta(2n)\Big(\frac{z}{\pi}\Big)^{2n},$$

so

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|.$$

In particular,

$$\zeta(2) = \pi^2|B_2| = \frac{\pi^2}{6}; \qquad \zeta(4) = \frac{\pi^4}{3}|B_4| = \frac{\pi^4}{90}; \qquad \zeta(6) = \frac{2\pi^6}{45}|B_6| = \frac{\pi^6}{945}.$$

By the way, the above formula for $\zeta(2n)$ gives a better way to determine the asymptotics of the Bernoulli numbers: since $\zeta(2n)\to 1$ as $n\to+\infty$, we must have

$$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}}$$

and thus, using Stirling's formula as before,

$$|B_{2n}| \sim 4\Big(\frac{n}{e\pi}\Big)^{2n}\sqrt{\pi n}.$$

7. *Categorification.*

The problem with such a structure type would be that some of the B_n are negative, and we don't know how to interpret a structure type which can be put on a set in a negative number of ways.

8. *Clay prize.*

I think I'll just forgo the \$ 1,000,000 and just claim one point of extra credit.