DEREK WISE
11 Feb. 2004
QUANTUM GRAVITY SEMINAR

Euler's Crazy Proof that $\sum_{n\in\mathbb{N}} n = -\frac{1}{12}$

1) A k-pointing of n is a function $k \to n$, and there are n^k of these. There are $n!$ orderings of n, so if a $P(k)$-str. on n is an ordering and k-pointing of n, then

$$|P(k)_n| = n!\,n^k.$$

2) $$|P(k)|(z) = \sum_{n=0}^{\infty} \frac{n!\,n^k z^n}{n!} = \sum_{n=0}^{\infty} n^k z^n.$$

3) $$|P(0)|(z) = \sum_{n=0}^{\infty} n^0 z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

4) An $N\Psi$-str. on S, i.e. an $A^*A\Psi$-str. on S, is:

A choice of $x \in S$ and an $A\Psi$-str. on $S - \{x\}$.

i.e.:

A choice of $x \in S$ and a Ψ-str. on $(S - \{x\}) + 1$

i.e.:

$\boxed{\text{A choice of } x \in S \text{ and a } \Psi\text{-str. on } S.}$

(Note: if we consider instead an $AA^*\Psi$-str. on S, we see that this amounts to a choice of $x \in S+1$ and a Ψ-str. on S. Then we can interpret an $[A, A^*]\Psi$-str. on S as a choice of any $x \in S+1$ such that $x \notin S$, and then a Ψ-str. on S. Since the "choice" of x is really no choice at all, this is the same as just a Ψ-str. on S. This gives us $[A, A^*] \cong 1$ as an isomorphism between operators on structure-types.)

5) An $NP(k)$-structure on S is a choice of $x \in S$, an ordering of S and a k-pointing $k \to S$. But a k-pointing $k \to S$ together with a choice of $x \in S$ is the same as a $k+1$ pointing $k+1 \to S$, where the additional element is mapped to x. Conversely, given $k+1 \to S$ and an ordering of S, we can call the image of the <u>last</u> point of S (according to the order) a "choice of $x \in S$" and get a k-pointing by restricting the domain of $k+1 \to S$ to k. This obviously gives an isomorphism of structure types:

$$NP(k) \cong P(k+1).$$

6) Decategorifying, we get

$$|A^* A P(k)|(z) = |P(k+1)|(z)$$

$$a^* a |P(k)|(z) = |P(k+1)|(z)$$

$$z \frac{d}{dz} |P(k)|(z) = |P(k+1)|(z)$$

Explicitly, this just says:

$$z \frac{d}{dz} \sum_{n=0}^{\infty} n^k z^n = \sum_{n=0}^{\infty} n^{k+1} z^n$$

or

$$z \sum_{n=0}^{\infty} n^{k+1} z^{n-1} = \sum_{n=0}^{\infty} n^{k+1} z^n$$

or

$$\sum_{n=0}^{\infty} n^{k+1} z^n = \sum_{n=0}^{\infty} n^{k+1} z^n$$

which is true!

7) $|P(1)|(z) = z \frac{d}{dz} |P(0)|(z)$

$$= z \frac{d}{dz} \frac{1}{1-z}$$

$$= z \cdot \frac{1}{(1-z)^2}$$

$$= \frac{z}{(1-z)^2}.$$

8) If we could evaluate $|P(1)|(z)$ at -1, we would get:

$$|P(1)|(-1) = \sum_{n=0}^{\infty} n(-1)^n = 0 - 1 + 2 - 3 + 4 - 5 + \cdots$$

But also:

$$|P(1)|(-1) = \frac{(-1)}{(1-(-1))^2} = \frac{-1}{4}, \text{ by \#7.}$$

So:

$$\text{"}1 - 2 + 3 - 4 + 5 - 6 + \cdots = \frac{1}{4}\text{"}.$$

9) $A \sum_{n=1}^{\infty} (-1)^{n+1} n = \frac{1}{4}$.

<u>PROOF:</u> The Taylor expansion of $\frac{z}{(1-z)^2}$ about $z=0$ agrees with the formal power series $|P(1)|(z)$, and converges for $|z|<1$. In particular, we have

$$\frac{-t}{(1-t)^2} = |P(1)|(-t) = \sum_{n=1}^{\infty} n(-t)^n$$

for $t \in [0,1)$. The left-hand side of this has limit

$\frac{-1}{(1-(-1))^2} = \frac{-1}{4}$ as $t \uparrow 1$. For the right hand side, we get

$$\lim_{t\uparrow 1} \sum_{n=1}^{\infty} n(-t)^n = -\lim_{t\uparrow 1} \sum_{n=1}^{\infty} t^n (-1)^n n$$

$$= -A\sum_{n=1}^{\infty} (-1)^{n+1} n,$$

by definition of the Abel sum. Hence $A\sum_{n=1}^{\infty}(-1)^{n+1}n = \frac{1}{4}$. ∎

10) $|P(2)|(z) = z\frac{d}{dz}|P(1)|(z)$

$$= z\frac{d}{dz}\frac{z}{(1-z)^2}$$

$$= z\left(\frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3}\right)$$

$$= z\,\frac{1+z}{(1-z)^3}$$

Since this function is analytic in the disc $|z|<1$, we have, for all points z in the disc:

$$\frac{z(1+z)}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^n$$

In particular, if $t \in [0,1)$,

$$-\frac{t(1+t)}{(1-t)^3} = -\sum_{n=1}^{\infty} n^2 t^n$$

and taking the limit as $t \uparrow 1$, we find

$$0 = A\sum (-1)^{n+1} n^2$$

so the alternating (Abel) sum of the squares of all positive integers is zero!

Finally, we use this to find the 'sum' of all the squared positive integers:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots = \underbrace{(1^2 - 2^2 + 3^2 - 4^2 + \cdots)}_{0} + (2\cdot 2^2 + 2\cdot 4^2 + 2\cdot 6^2 + \cdots)$$

$$= 2^3(1^2 + 2^2 + 3^2 + 4^2 + \cdots)$$

or

$$(1-2^3)(1^2 + 2^2 + 3^2 + 4^2 + \cdots) = 0$$

$$\Rightarrow \boxed{(1^2 + 2^2 + 3^2 + 4^2 + \cdots) = 0}$$

and indeed, the Riemann zeta function has zeros at all of the negative even integers!

✓