From the Associator to the $6j$ Symbols

John C. Baez, January 20, 2005

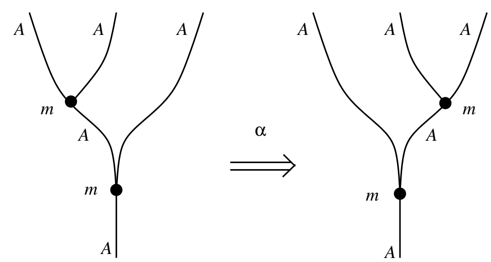
Given a 2-algebra A with multiplication

$$m\colon A \otimes A \to A,$$

the associator is a natural isomorphism

$$\alpha\colon (m \otimes 1)m \Rightarrow (1 \otimes m)m$$

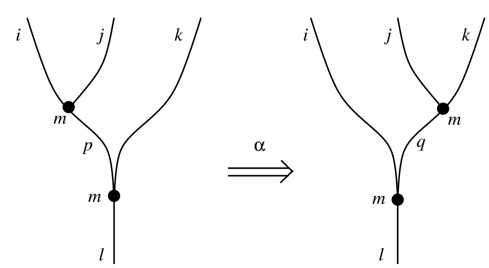
which we can draw as this process:



If we pick a basis of objects e^i for the 2-vector space A and define vector spaces

$$m^{ij}_k = \hom(e^k, e^i \otimes e^j),$$

we can label all the strands in the above diagram by basis objects:



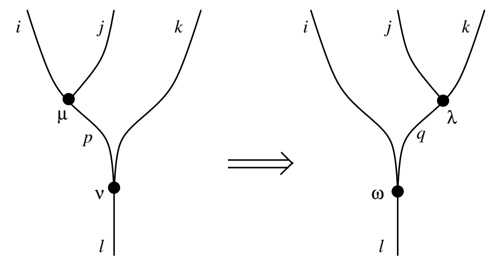
and use this to read off how the associator gives a bunch of linear operators:

$$\alpha\colon m^{ij}_p \otimes m^{pk}_l \to m^{jk}_q \otimes m^{iq}_l .$$

If we then go ahead and pick bases of all four vector spaces here:

$$E^\mu \in m^{ij}_p, \quad E^\nu \in m^{pk}_l, \quad E^\lambda \in m^{jk}_q, \quad E^\omega \in m^{iq}_l,$$

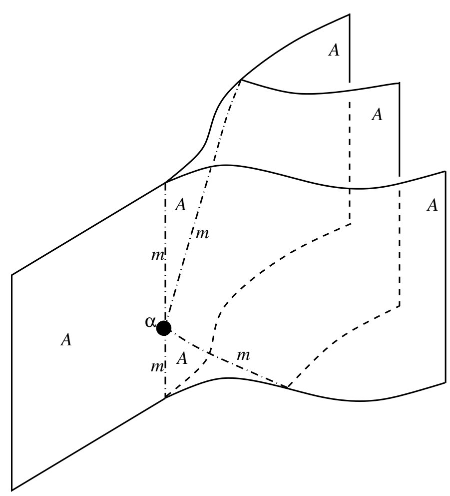
we can describe the above linear operators as matrices! Note that picking bases of these vector spaces amounts to labelling the *vertices* in the above diagram by basis *vectors*:



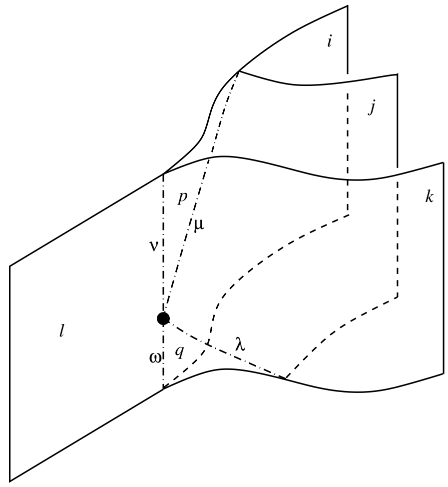
Thus, we can describe the associator in a very low-brow way as a bunch of numbers depending on the 6 Latin indices i, j, k, l, p, q and the 4 Greek indices $\mu, \nu, \lambda, \omega$. This is a categorified version of the usual index notation that physicists like — but now there are two layers of indices: Latin indices for bases of 2-vector spaces, and Greek indices for bases of vector spaces!

In the special case where $A = \mathrm{Rep}(\mathrm{SU}(2))$ these numbers are called the **6*j* symbols**, since then the spaces m_k^{ij} are at most 1-dimensional, which allows us to ignore the Greek indices above, leaving a number that depends on 6 Latin indices.

For applications to physics and topology, it's good to imagine the associator as a process that sweeps out a surface in spacetime:



Our index notation then amounts to labelling all the *faces* of this surface by *Latin letters* corresponding to a basis of our *2-vector space A*, and labelling all the *edges* of this surface by *Greek letters* corresponding to bases of various *vector spaces.* So, we get a number from this picture:



It’s also crucial to note that the 2d surface traced out by the associator is Poincaré dual to a tetrahedron. We can draw it like this: