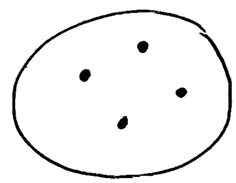
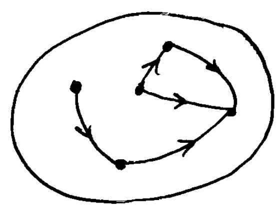
GAUGE THEORY & TOPOLOGY

(Winter 2005)

Categorification is the "process" of replacing set-based math:



by category based math:



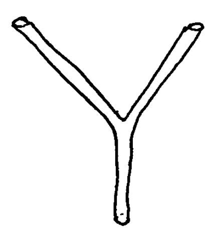
and it has a tendency to "boost dimensions." Last quarter we built 2d TQFTs from certain nice monoids (namely, semisimple algebras); now we'll categorify this & build 3d TQFTs from certain nice monoidal categories ("semisimple 2-algebras")

We saw last quarter that a great example of a semisimple algebra is the group algebra $\mathbb{C}[G]$ for a finite group G; we'll see that these give 2d TQFTs where the partition function is computed as a path integral over the space of "G-bundles with flat connection." (I.e. we get a simple sort of gauge theory with G as its gauge group) Similarly, a great example of a semisimple 2-algebra is Vect$[G]$ for a finite group G; these give

3d TQFTs whose partition function can be computed the same way.

We can also let G be a Lie group, but then the partition function often diverges, but the resulting "near-TQFT" is very interesting & it's called "BF theory." This theory is like the harmonic oscillator or free quantum field theory — an "exactly soluble" theory serving as a springboard for more interesting theories.

In 2d, for example, Yang-Mills theory can be described as a perturbed BF theory. (Also in higher dimensions, but it gets a lot harder!) In 3d, quantum gravity without matter is a BF theory with $G = SO(2,1)$. Recently, Freidel & Louapre have described 3d quantum gravity with matter as a BF theory. on a 3d manifold with "tubes" removed:



Also, Freidel & Starodubtsev have described 4d gravity as a perturbed BF theory with $G = SO(3,2)$.

To begin...

Our work on 2d TQFTs was based on linear algebra, done via diagrams — i.e. the (symmetric monoidal) category Vect. Now, to get higher-dimensional diagrams, we'll categorify

Vect & get a (symmetric monoidal) 2-category 2Vect.
We can do this starting either from the low-brow or high-brow definition of Vect:

Vect_{lb} has as <u>objects</u> $\mathbb{C}^n$ $(n = 0, 1, \ldots)$

& has as <u>morphisms</u> $f: \mathbb{C}^n \to \mathbb{C}^m$ $n \times m$ complex matrices

Vect_{hb} has as <u>objects</u> finitely generated modules of the ring $\mathbb{C}$

& has as <u>morphisms</u> linear operators

It's easy to check that there's an inclusion (i.e. a functor which is full & faithful):

$$\mathrm{Vect}_{lb} \hookrightarrow \mathrm{Vect}_{hb}$$

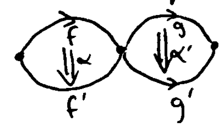
but the first big theorem of linear algebra says every highbrow vector space is isomorphic to some $\mathbb{C}^n$, which says this functor is essentially surjective. Thus we see Vect_{lb} & Vect_{hb} are equivalent as categories. (In fact Vect_{lb} is a skeleton of Vect_{hb}: a full subcategory containing one representative of each isomorphism class.)

Let's categorify Vect_{lb}:

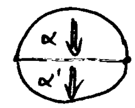
$2\mathrm{Vect}_{lb}$ has as <u>objects</u> Vect^n $(n = 0, 1, \ldots)$ $(\mathrm{Vect} = \mathrm{Vect}_{lb}$ or $\mathrm{Vect}_{hb})$

has as <u>morphisms</u> $f: \mathrm{Vect}^n \to \mathrm{Vect}^m$ $n \times m$ matrices of vector spaces

& has as <u>2morphisms</u> $\mathrm{Vect}^n \underset{g}{\overset{f}{\Downarrow \alpha}} \mathrm{Vect}^m$ $n \times m$ matrices of linear operators

Composing morphisms is done by matrix multiplication, but with $\oplus$ & $\otimes$ replacing $+$ & $\times$. Horizontal composition of 2-morphisms works the same way: 

$\alpha \circ \alpha' : fg \Rightarrow f'g'$

Vertical composition of 2-morphisms: 

is done by composing linear operators entrywise

$$\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}\begin{pmatrix} g & g' \\ g'' & g''' \end{pmatrix} = \begin{pmatrix} fg & f'g' \\ f''g'' & f'''g''' \end{pmatrix}$$

$2\mathrm{Vect}_{hb}$ has as <u>objects</u> <u>2-vector spaces</u>, i.e. <u>$\mathbb{C}$-linear</u> <u>finitely generated</u> <u>semisimple</u> <u>abelian</u> categories

- $\mathbb{C}$-linear: $\hom(x,y)$ is a complex vector space & composition is bilinear: $\circ : \hom(x,y) \otimes \hom(y,z) \to \hom(x,z)$
- finitely generated: there are finitely many isomorphism classes of simple objects.
- semisimple: every object is a direct sum of <u>simple</u> ones e_i: $\hom(e_i, e_i) \cong \mathbb{C}$
- abelian: category with $\oplus$, kernels, cokernels, satisfying a list of axioms

has as <u>morphisms</u> <u>exact</u> <u>$\mathbb{C}$-linear</u> functors

- exact: our functor F preserves $\oplus$, kernels, and cokernels
- $\mathbb{C}$-linear: our functor F is linear on hom-spaces: $F : \hom(x,y) \to \hom(Fx, Fy)$ is linear

has as <u>2-morphisms</u> natural transformations.

<u>Thm (Yetter)</u>: There's a 2-functor

$$2\mathrm{Vect}_{eb} \hookrightarrow 2\mathrm{Vect}_{hb}$$

which is a 2-equivalence, & in fact $2\mathrm{Vect}_{eb}$ is a kind of skeleton of $2\mathrm{Vect}_{hb}$.

Let's reemphasize a point from week 10 of the fall quarter:

The passage from Vect to 2Vect is analogous to & relies on the passage from $\mathbb{C}$ to Vect. Just as $\mathbb{C}$ is a commutative rig with

$$+,\ \times,\ 0,\ 1,$$

Vect is a "symmetric 2-rig" with

$$\oplus,\ \otimes,\ \{0\},\ \mathbb{C}$$

with all the usual commutative rig laws holding up to specified isomorphism. A bit more precisely:

$(\text{Vect}, \oplus, \{0\})$ is a symmetric monoidal category ← categorification of fact that $(\mathbb{C}, +, 0)$ is a commutative monoid

$(\text{Vect}, \otimes, \mathbb{C})$ is a symmetric monoidal category

& $\otimes$ distributes over $\oplus$ up to the <u>distributor</u>:

$$d_{U,V,W} : U \otimes (V \oplus W) \xrightarrow{\sim} (U \otimes V) \oplus (U \otimes W)$$

satisfying some extra coherence laws discovered by Kelley and Laplaza.

Just as a vector space is a $\mathbb{C}$-module, a 2-vector space is a "Vect-module": e.g. given $(V_1, \ldots, V_n) \in \text{Vect}^n$ and $C \in \text{Vect}$ we can multiply the former by the latter:

$$C \otimes (V_1, \ldots, V_n) = (C \otimes V_1, \ldots, C \otimes V_n)$$

Also, just as we can direct-sum and tensor product vector spaces, we can do so with 2-vector spaces:

$$\mathrm{Vect}^n \oplus \mathrm{Vect}^m \simeq \mathrm{Vect}^{n+m}$$

- Vect^n: pick a basis $e_i = (\{0\}, \dots, \mathbb{C}, \dots, \{0\})$ (↑ ith slot)
- Vect^m: pick a basis f_j
- Vect^{n+m}: has basis $\{e_i, f_j\}$

$$\mathrm{Vect}^n \otimes \mathrm{Vect}^m \simeq \mathrm{Vect}^{nm}$$

- Vect^{nm}: has basis $\{e_i \otimes f_j\}$

If we have an (associative, unital) algebra A, i.e. a monoid in Vect, it has addition

$$+ : A \oplus A \longrightarrow A$$

direct sum is just the thing you need to describe addition as an operator

similarly for $\otimes$, $\times$

& multiplication

$$\times : A \otimes A \longrightarrow A$$

(examples of the "microcosm principle" – "as above, so below")

We can describe multiplication using structure constants m^i_{jk} if we pick a basis $e_i \in A$:

$$e_j e_k = \sum_i m^i_{jk} e_i \qquad m^i_{jk} \in \mathbb{C}$$

Similarly, a "2-algebra" A is a monoidal category in ~~2Vect~~

2 Vect will have addition and multiplication

$$+ : A \oplus A \longrightarrow A$$
$$\times : A \otimes A \longrightarrow A$$

now relying on $\oplus$ & $\otimes$ in 2Vect!

Simplest example of an algebra: $\mathbb{C}$
Simplest example of a 2-algebra: Vect

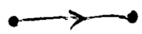
We can describe multiplication in a 2-algebra A using "structure constants" in terms of a basis of simple objects $e_i \in A$:

$$e_j e_k = \bigoplus_i m^i_{jk} \otimes e_i \qquad m^i_{jk} \in \mathrm{Vect}$$

BUILDING 2D VS 3D TQFTS

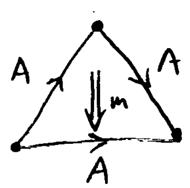
2D

unlabelled



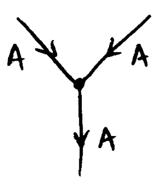
$A \in \mathrm{Vect}$

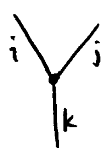
or dually:



$m : A \otimes A \to A$

or dually:





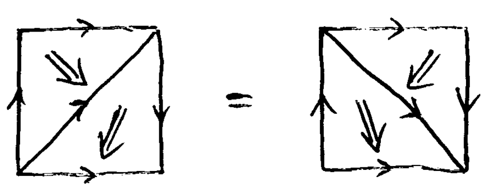
$= m^{ij}_k \in \mathbb{C}$

where

$$e^i e^j = \sum_k m^{ij}_k e^k$$

for the basis $e^i \in A$

2-2 move



associative law for m:

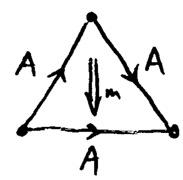
$$(m \otimes 1)m = (1 \otimes m)m$$

3D

unlabelled

$A \in 2\mathrm{Vect}$

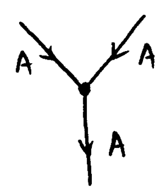
or dually:



$m : A \otimes A \to A$

└ morphism in 2Vect
– an exact functor

or dually:

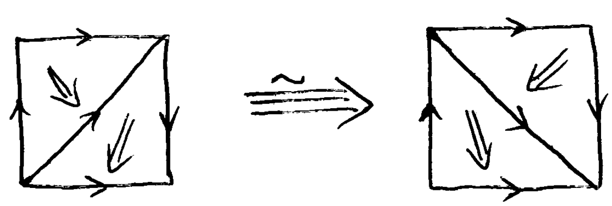


$= m^{ij}_k \in \mathrm{Vect}$

where

$$e^i e^j = \sum m^{ij}_k \otimes e^k$$

for the basis $e^i \in A$.



associator for m:

$$\alpha : (m \otimes 1)m \xRightarrow{\sim} (1 \otimes m)m$$