11 Jan 2005

Let's back off from constructing 3d TQFTs & look at:

DIAGRAMS FOR CATEGORIFIED LINEAR ALGEBRA

Given a morphism in Vect, $T: V \to W$, we can get a matrix of numbers $T^i_j \in \mathbb{C}$ describing it by picking bases

$$e^i \in V \qquad f^j \in W$$

We have:

$$T^i_j = f_j(Te^i)$$

where $f_j \in W^*$ lies in the dual basis: $\qquad W^* = \hom(W, \mathbb{C})$

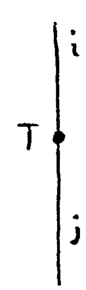
$$f_j(f^i) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Conversely, given T^i_j we can get $T: V \to W$ by

$$Te^i = \sum_j T^i_j f^j$$

In short we have 2 equivalent viewpoints:

morphism



$:= T^i_j \in \mathbb{C}$

like a transition amplitude in QFT

number

Now let's categorify this:

Given a morphism in 2Vect, $T : V \longrightarrow W$, we can get a matrix of vector spaces $T^i_j \in \mathrm{Vect}$ describing it by picking bases of <u>simple</u> objects $e^i \in V$, $f^j \in W$.

(Note: for <u>simple</u> objects $\hom(e^i, e^j) \cong \begin{cases} \mathbb{C} & i = j \\ 0 & i \neq j \end{cases} = \delta^{ij}$)

We have:

$$T^i_j = f_j(Te^i)$$

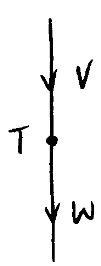
where $f_j \in W^* = \hom(W, \mathrm{Vect})$ is the dual basis

$$f_j(f^i) = \delta^i_j = \begin{cases} \mathbb{C} & i = j \\ \{0\} & i \neq j \end{cases}$$

Conversely, given $T^i_j \in \mathrm{Vect}$ we can get $T : V \to W$ by

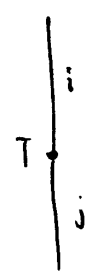
$$Te^i = \bigoplus_j T^i_j \otimes f^j$$

In short, we have 2 equivalent viewpoints:



morphism

$\sim$



$:= T^i_j \in \mathrm{Vect}$

vector space

Just one difference: in 2Vect, unlike Vect, we can express $e_j \in V^*$ in terms of $e^j \in V$ as follows

$$e_j(v) \cong \hom(e^j, v) \qquad \in \mathrm{Vect}$$

$e^j = (0, \ldots, 1, \ldots, 0)$
$v = (v_1, \ldots, v_j, \ldots, v_n)$

note: $\hom(\ ,\ )$ is like an inner product

In particular:

$$e_j(e^k) \cong \hom(e^j, e^k) \cong \delta_j^k = \begin{cases} \mathbb{C} & j = k \\ 0 & j \neq k \end{cases}$$

In short, a 2-vector space is a bit like a <u>Hilbert</u> space with the "inner product" being $\hom(-,-)$. In fact

$$\hom(-,-) : V^{op} \otimes V \longrightarrow \mathrm{Vect}$$

gives an equivalence

$$V^{op} \simeq V^*$$
$$v \longmapsto \hom(v,-)$$

In QM,

$$\langle \psi, \phi \rangle = \text{amplitude for the state } \psi \text{ to be (measured as) } \phi$$

In category theory

$$\hom(x,y) = \text{set of ways to get from } x \text{ to } y$$

& in 2Vect

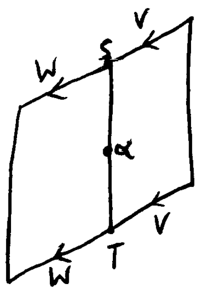
$$\hom(x,y) = \text{vector space of ways to go from } x \text{ to } y$$

But more exciting is the fact that 2Vect has 2-morphisms...

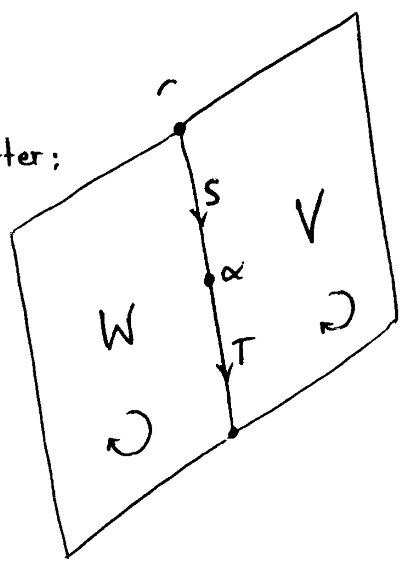
2-morphisms in 2Vect are natural transformations:

$$
\begin{tikzcd}
V \arrow[r, bend left=60, "S"] \arrow[r, bend right=60, "T"'] \arrow[r, phantom, "\alpha\Downarrow"] & W
\end{tikzcd}
$$

There should be 2 equivalent viewpoints: an "abstract" viewpoint (α is a natural transformation) and a "concrete" one (in terms of matrices). The abstract one is easy: just apply Poincare duality to $\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$ and rotate it to get a "2d Feynman diagram."



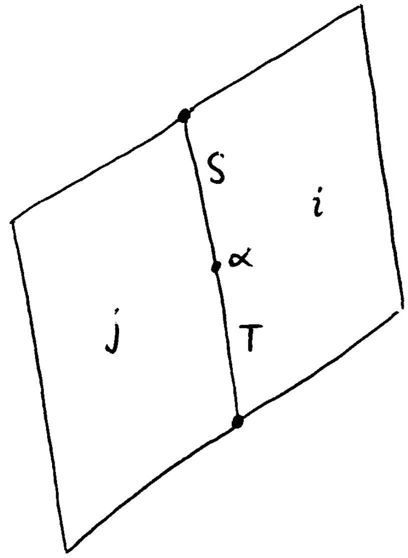
or better:



To turn this "abstract" picture into one involving matrices, first pick bases $e^i \in V$, $f^j \in W$. This gives matrices of vector spaces S^i_j, T^i_j; α should give a matrix of operators between these, say

$$\alpha^i_j : S^i_j \to T^i_j$$

we could draw this as:



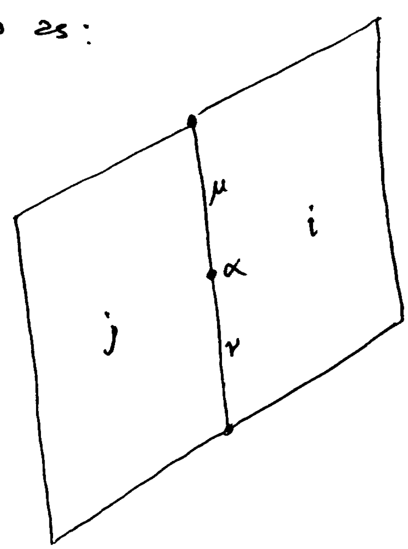
– i.e. replace V & W by labels for bases of them.
But we can go further & describe the operator α^i_j as a matrix in terms of bases for the vector spaces S^i_j & T^i_j. Pick bases:

$$E^\mu \in S^i_j \qquad F^\nu \in T^i_j$$

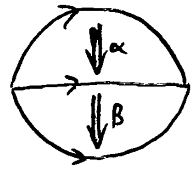
where μ, ν range over sets depending on i, j. Then α^i_j can be described using a matrix:

$$(\alpha^i_j)^\mu_\nu = F_\nu(\alpha^i_j(E^\mu)) \in \mathbb{C}$$

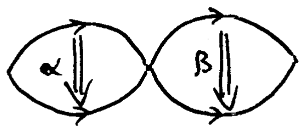
which we can draw as:



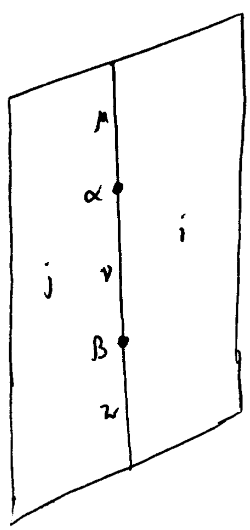
We can vertically:



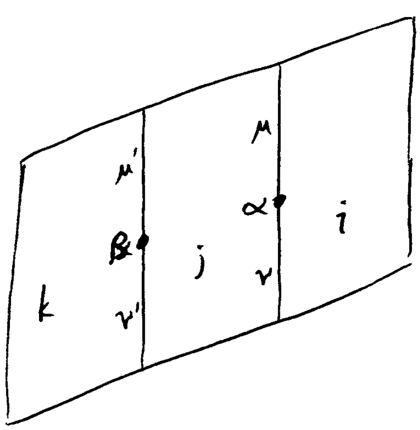
& horizontally



compose natural transformations as follows:



$$:= \sum_{\nu} (\alpha^{i}_{j})^{\mu}_{\nu} (\beta^{i}_{j})^{\nu}_{w}$$



$$:= \sum_{j} (\alpha^{i}_{j})^{\mu}_{\nu} (\beta^{j}_{k})^{\mu'}_{\nu'}$$

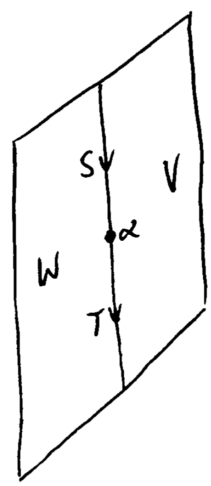
13 Jan 2005

DIAGRAMS FOR CATEGORIFIED LINEAR ALGEBRA (cont.)

In the highbrow description, a 2-morphism in 2Vect is a natural transformation between exact functors:

$$\begin{tikzcd} V \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\alpha"] & W \end{tikzcd}$$

or as a "spin foam" picture:



From this we get a matrix of linear operators α^i_j if we pick bases $e^i \in V$, $f^j \in W$. S & T give matrices of vector spaces:

$$S^i_j = \hom(f^j, Se^i)$$

$$T^i_j = \hom(f^j, Te^i)$$

The natural transformation α gives for each object $e^i \in V$, a morphism

$$\alpha_{e^i} : S(e^i) \longrightarrow T(e^i)$$

and composition with this gives the linear operator

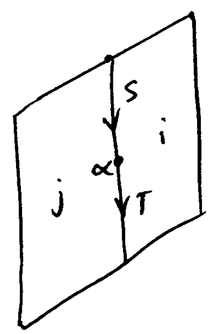
$$\alpha^i_j : \hom(f^j, Se^i) \longrightarrow \hom(f^j, Te^i)$$

i.e.

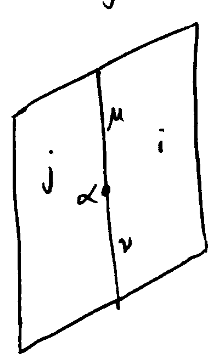
$$\alpha^i_j : S^i_j \to T^i_j$$

— the middlebrow description of α as a matrix of operators.

Our spin foam notation for this is:



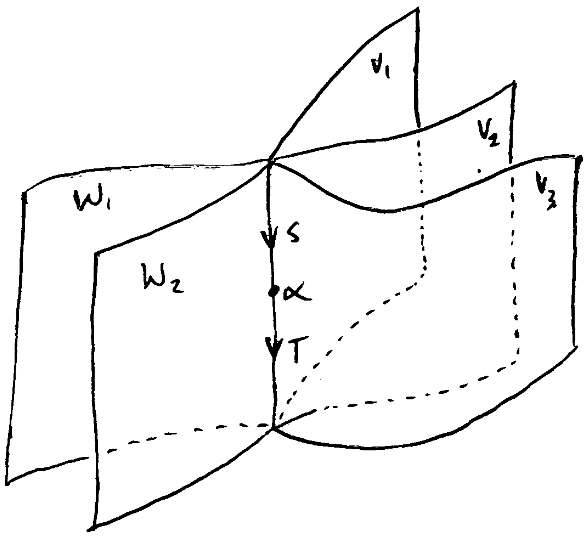
For a truly lowbrow description of α, pick bases $E^\mu \in S^i_j$ & $F^\nu \in T^i_j$ & write α as a matrix of matrices w.r.t. the bases. Pictorially:



This whole story should continue for 3-vector spaces, 4-vector spaces, etc... giving us higher-dimensional "membrane" pictures of higher dimensional linear algebra — possibly important in higher-dim TQFTs, M-theory, etc...

All this stuff applies to fancier situations:

$$\begin{tikzcd} V_1 \otimes \cdots \otimes V_n \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow \alpha"] & W_1 \otimes \cdots \otimes W_m \end{tikzcd}$$



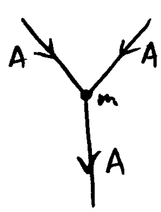
Now let's return to our main motivating example. Suppose A is a "2-algebra" — i.e., a 2-vector space with multiplication

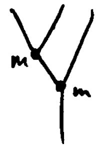
$$m : A \otimes A \longrightarrow A$$

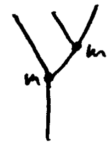
making A into a monoidal category, with associator:

$$\alpha : (m \otimes 1) m \xRightarrow{\sim} (1 \otimes m) m$$

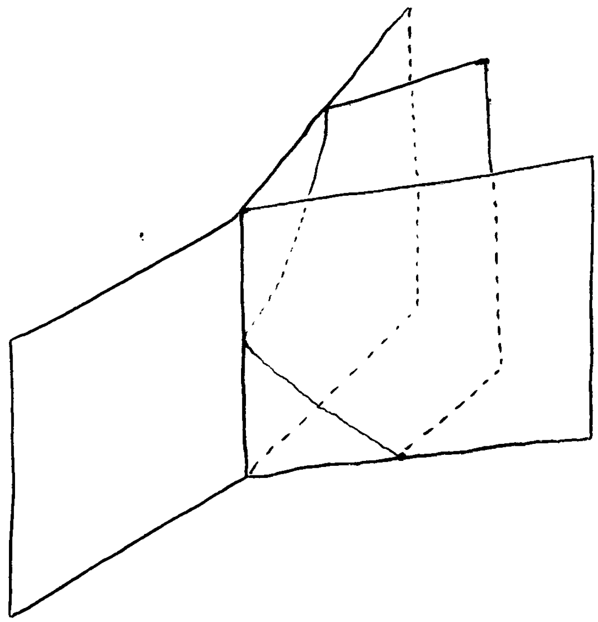
or pictorially:

$m =$ 

$(m\otimes 1)m =$ 

$(1\otimes m)m =$ 

$\alpha =$



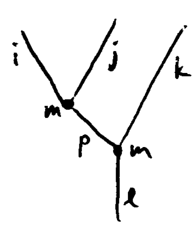
Let's turn this into a matrix of matrices (or "tensor of tensors").
First pick a basis e^i of A, & write

$$m_k^{ij} = \hom(e^k, m(e^i, e^j))$$

or, using the usual notation $m(e^i, e^j) = e^i \otimes e^j$,

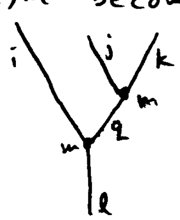
$$m_k^{ij} = \hom(e^k, e^i \otimes e^j).$$

Then $(m\otimes 1)m$ becomes:



$$= \bigoplus_p m_p^{ij} \otimes m_\ell^{pk}$$

and $(1\otimes m)m$ becomes:



$$= \bigoplus_q m_\ell^{iq} \otimes m_q^{jk}$$

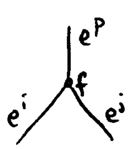
& α should give a linear operator from the first to the second. α starts out life as a natural transformation, giving

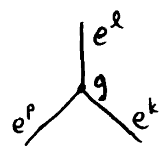
$$\alpha_{e^i,e^j,e^k} : (e^i \otimes e^j) \otimes e^k \xrightarrow{\sim} e^i \otimes (e^j \otimes e^k)$$

$$\begin{array}{rcl}
\bigoplus\limits_p m_p^{ij} \otimes m_\ell^{pk} & = & \bigoplus\limits_p \hom(e^p, e^i \otimes e^j) \otimes \hom(e^\ell, e^p \otimes e^k) \\
& & \wr\!\parallel \\
& & \hom(e^\ell, (e^i \otimes e^j) \otimes e^k) \\
& & \Big\downarrow \text{ compose w. } \alpha_{e^i,e^j,e^k} \\
& & \hom(e^\ell, e^i \otimes (e^j \otimes e^k)) \\
& & \wr\!\parallel \\
\bigoplus\limits_q m_\ell^{iq} \otimes m_q^{jk} & = & \hom(e^\ell, e^i \otimes e^q) \otimes \hom(e^q, e^j \otimes e^k)
\end{array}$$

The first isomorphism comes from this map:

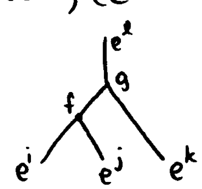
$$\hom(e^p, e^i \otimes e^j) \otimes \hom(e^\ell, e^p \otimes e^k)$$





$$\Big\downarrow$$

$$\hom(e^\ell, (e^i \otimes e^j) \otimes e^k)$$



& in fact:

$$\hom(e^\ell, (e^i \otimes e^j)\otimes e^k)$$

$$\Vert\wr \qquad\qquad e^i \otimes e^j = \bigoplus_p m_p^{ij} \otimes e^p$$

$$\bigoplus_p \hom(e^\ell, m_p^{ij} \otimes e^p \otimes e^k)$$

$$\Vert\wr$$

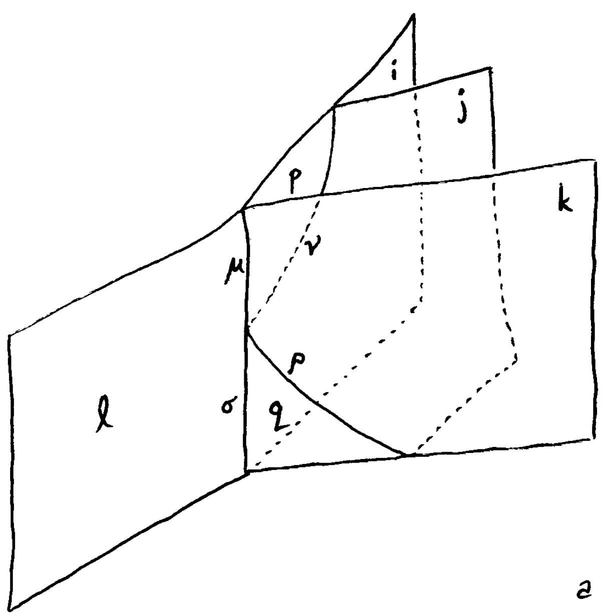
$$\bigoplus_p m_p^{ij} \otimes \hom(e^\ell, e^p \otimes e^k)$$

$$\Vert \qquad\qquad \text{def of } m_p^{ij}$$

$$\bigoplus \hom(e^p, e^i \otimes e^j) \otimes \hom(e^p, e^i \otimes e^j)$$

The second iso. is the same sort of thing.

So: we get a matrix of linear operators, & then a matrix of matrices:



6 planes: i, j, k, ℓ, p, q meeting along 4 edges: μ, ν, σ, ρ

For example, μ labels a basis of the v.s. m_ℓ^{pk}

This picture with 6 planes, meeting 3 at a time along 4 edges, is secretly the Poincaré dual 2-skeleton of a tetrahedron...

The tetrahedron has 6 edges, 3 surrounding each of the 4 triangular faces: