18 January 2005

BUILDING 3D TQFTs

Suppose $A \in 2\mathrm{Vect}$ is a 2-algebra. We want to build a 3d TQFT from it: a symmetric monoidal functor

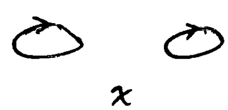
$$Z : 3\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

In fact, we'll do much more: we'll build an <u>extended</u> 3d TQFT, which assigns algebraic data to 1-, 2-, & 3-dimensional manifolds: a symmetric monoidal 2functor

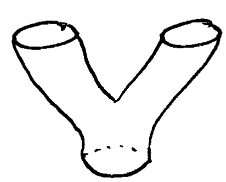
$$Z : 3\mathrm{Cob}_2 \longrightarrow 2\mathrm{Vect}$$

where $3\mathrm{Cob}_2$ is a (symmetric monoidal) 2-category with

0) (compact oriented) 1-manifolds as objects:

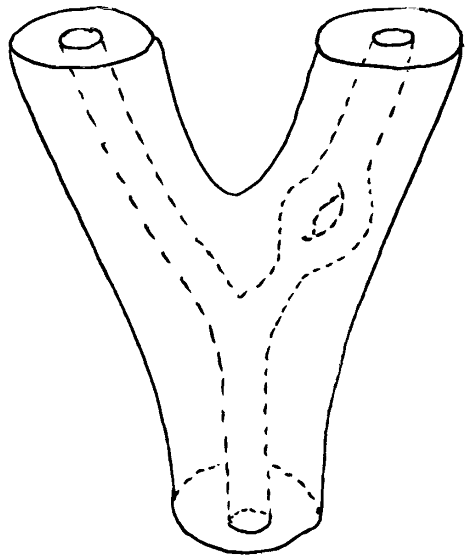


1) (compact oriented) 2-dimensional cobordisms between these as morphisms:



duals ~ orientation reversal
(algebra) (geometry)

2) (compact oriented) 3-dimensional cobordisms between those as 2-morphisms:



$$\begin{tikzcd} X \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "g"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\ Y \end{tikzcd}$$

Note: this is a manifold with corners

Such an <u>extended</u> TQFT automatically gives an ordinary TQFT. To begin to see this, note:

0) If $x \in 3\mathrm{Cob}_2$ is the empty set $x = \emptyset$, then

$$Z(x) \simeq \mathrm{Vect}$$

because $\emptyset \in 3\mathrm{Cob}_2$ is the unit object for disjoint union (the $\otimes$ in $3\mathrm{Cob}_2$) while Vect is the unit for the $\otimes$ in 2Vect.

1) If $f : \emptyset \longrightarrow \emptyset$ is a morphism in $3\mathrm{Cob}_2$ – i.e. a 2-manifold with empty boundary (a "closed" 2-manifold) – then our extended TQFT gives

$$\begin{array}{ccccc} Z(f) : & Z(\emptyset) & \longrightarrow & Z(\emptyset) \\ & \wr\wr & & \wr\wr \\ & \mathrm{Vect} & & \mathrm{Vect} \end{array}$$

i.e. a 1×1 matrix of vector spaces, i.e. a vector space!

Good: this is just what an ordinary 3d TQFT would do!

2) If we have a 2-morphism

$$\begin{tikzcd} \emptyset \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "g"] \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] \\ \emptyset \end{tikzcd}$$

in $3Cob_2$ (i.e. a cobordism between closed 2-manifolds)

our extended TQFT gives

$$Z(\theta) : Z(f) \Longrightarrow Z(g)$$

i.e. a 1×1 matrix of linear operators, i.e. an <u>operator</u>!

Good: this is what an ordinary 3d TQFT would do.

Summary: we have

$$\hom_{3Cob_2}(\emptyset, \emptyset) \simeq 3Cob$$

a hom-category in the 2-category $3Cob_2$

AND:

$$\hom_{2Vect}(Vect, Vect) \simeq Vect$$

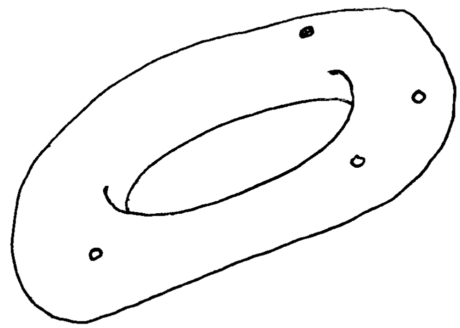
& our extended TQFT

$$Z : 3Cob_2 \longrightarrow 2Vect$$

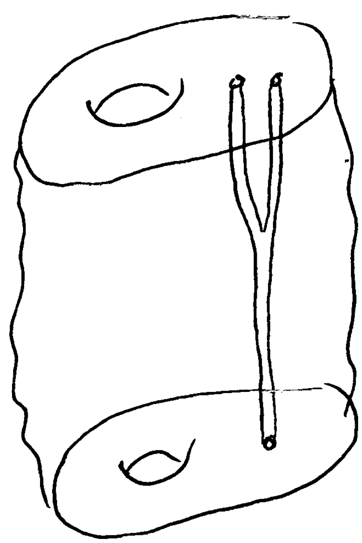
<u>restricts</u> to an ordinary TQFT

$$Z : 3Cob \longrightarrow Vect.$$

In physics, the fact that our extended TQFT assigns data to 2-manifolds <u>with boundary</u>:



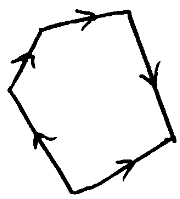
allows us to describe 2d spaces <u>including</u> point particles, which can undergo various properties as time passes:



(although here we need some more sophisticated 2category theory than we've talked about so far: We need 2-morphisms $\theta : f \Rightarrow g$ where $f : x \to y$, $g : x' \to y'$ — requiring some enhancements of our formalism

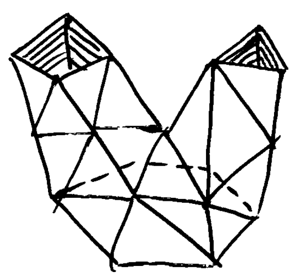
But: How do we get our 3d extended TQFT from a 2-algebra $A \in \mathrm{Vect}$? We'll start by triangulating everything in sight, & attempting to define:

0) an object $\tilde{Z}(x) \in 2\mathrm{Vect}$ for every triangulated 1-manifold x:



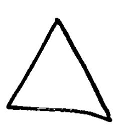
$$\overset{\tilde{Z}}{\longmapsto} \; A^{\otimes 5}$$

1) a morphism $\tilde{Z}(f) : \tilde{Z}(x) \longrightarrow \tilde{Z}(y)$ in $2\mathrm{Vect}$ for every triangulated 2d cobordism

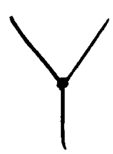


$$\begin{tikzcd} x \arrow[d, "f"'] & & \tilde{Z}(x) \arrow[d, "\tilde{Z}(f)"] \\ y & \arrow[r, mapsto, "\tilde{Z}"] & \tilde{Z}(y) \end{tikzcd}$$

We'll do this by our old "Feynman diagram" or "Spin network" trick:



$\rightsquigarrow$



$\rightsquigarrow$

$$\begin{tikzcd} A \otimes A \arrow[d, "m"] \\ A \end{tikzcd}$$

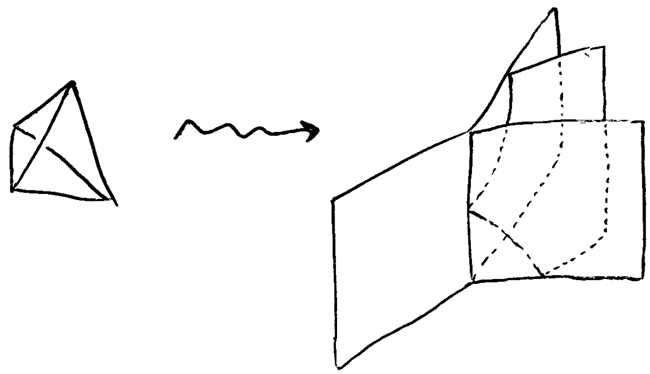
2) a 2-morphism

$$\begin{tikzcd} \tilde{Z}(x) \arrow[r, bend left, "\tilde{Z}(f)"] \arrow[r, bend right, "\tilde{Z}(g)"'] \arrow[r, phantom, "\Downarrow \tilde{Z}(\theta)"] & \tilde{Z}(y) \end{tikzcd} \quad \text{in } 2\mathrm{Vect}$$

for every triangulated 3d cobordism

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \theta"] & y \end{tikzcd}$$

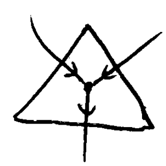
We'll do this by our new "spin foam" trick:



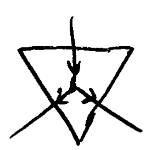
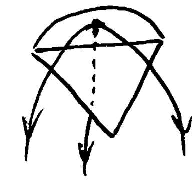
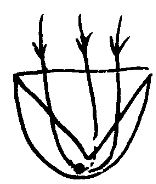
$$\rightsquigarrow \quad \begin{array}{c} (m\otimes 1)m \\ \Big\Downarrow \alpha \\ (1\otimes m)m \end{array}$$

There are 2 main obstacles in our way.

1) As in the 2d case, we need to define $\tilde{Z}$ not just on



but also:

  & 

As in the 2d case, this involves <u>duality</u>: need a pairing:

$$g : A \otimes A \longrightarrow \mathrm{Vect}$$

giving:

$$A \xrightarrow{\sim} A^* \cong \hom(A, \mathrm{Vect})$$
$$a \longmapsto g(a, -)$$

Here A will need to be semisimple. Note every 2-vector space A has:

$$\begin{aligned} \hom : A^{op} \otimes A &\longrightarrow \mathrm{Vect} \\ a \otimes b &\longmapsto \hom(a,b) \end{aligned}$$

which gives

$$\begin{aligned} A^{op} &\xrightarrow{\sim} A^* \\ a &\longmapsto \hom(a,-) \end{aligned} \qquad \text{(note: \underline{not} } A \simeq A^*\text{)}$$

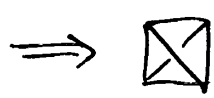
<u>Also</u> we need some ability to rotate our tetrahedra, getting not just

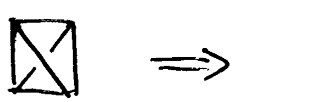
1 

but also

2 

3 

4 

5 

This requires another kind of duality. Both levels of duality are related to A being "semisimple".

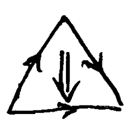
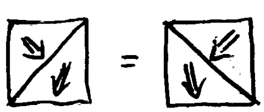
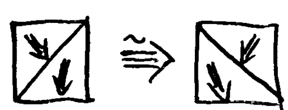
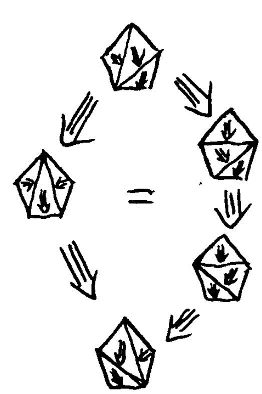
2) $\tilde{Z}$ should have some triangulation independence: we need the 2-3 & 1-4 Pachner moves. The 2-3 move is the pentagon identity. The 1-4 is related to semisimplicity.

20 Jan 2005

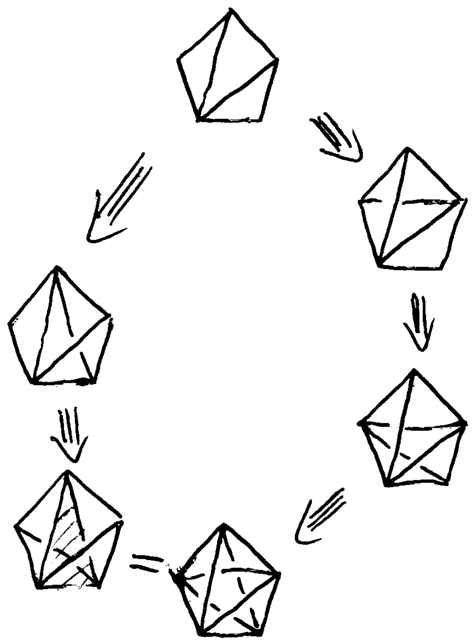
Let's continue trying to build a 3d extended TQFT:

$$Z : 3\mathrm{Cob}_2 \longrightarrow 2\mathrm{Vect}$$

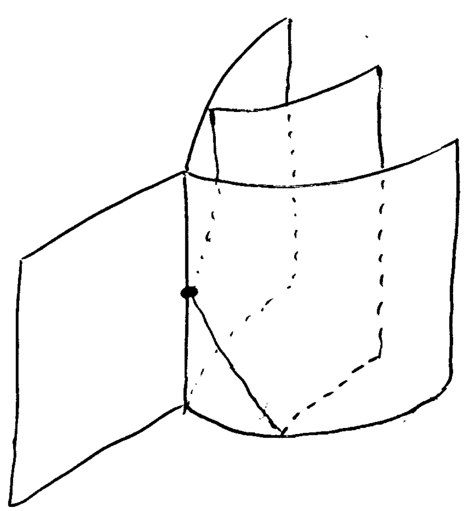
from a 2-algebra $A \in 2\mathrm{Vect}$ according to our plan, but only addressing a few interesting issues (no complete treatment along these lines exists yet).
Let's review our chart showing the analogy to the 2d case:

2d		3d	
•		•	
	$A \in \mathrm{Vect}$		$A \in 2\mathrm{Vect}$
	$m : A \otimes A \to A$		$m : A \otimes A \longrightarrow A$
	associative law for m: A is an algebra		associator for m $\alpha : (m \otimes 1)m \Rightarrow (1 \otimes m)m$
			pentagon identity for α: A is a 2-algebra

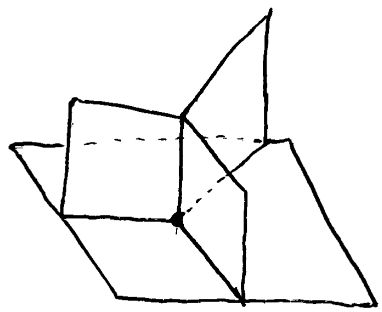
We can draw this in various other styles. E.g., as "2-3 move" going from the front to the back of a 4-simplex:



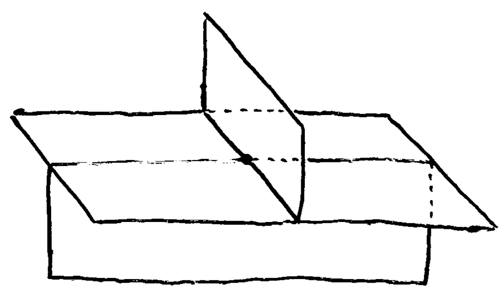
Or: the Poincaré dual "spin foam" picture, where the associator looks like



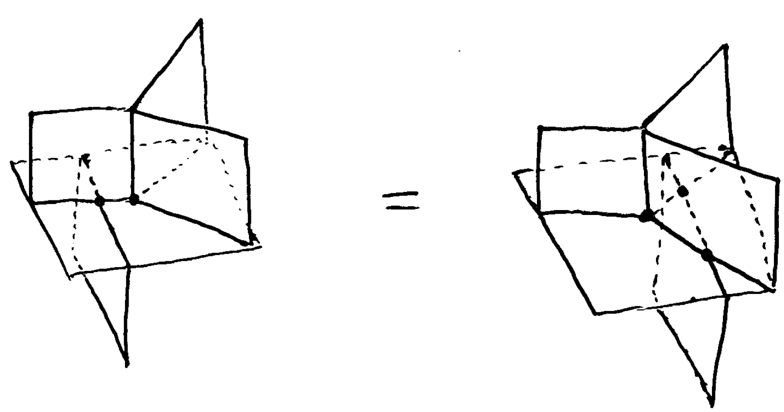
All that matters here is that we have 6 faces, 3 meeting along each of 4 edges that meet at a vertex. Two other ways to draw it:



or



These styles let us draw the 2-3 move as a move on spin foams:



(2 vertices) = (3 vertices)

If we labelled all faces with indices for a basis of A & edges with indices for bases of

$$\hom(e^i, e^j \otimes e^k) = m_i^{jk}$$

these diagrams specify two equal numbers – which for $A = \mathrm{Rep}(SU(2))$ are called the Biedenhorn-Elliot identities.

2d

We need A s.t. this pairing:

(the Killing pairing)

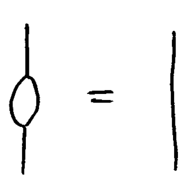
is nondegenerate:

$$A \otimes A \xrightarrow{g} \mathbb{C}$$

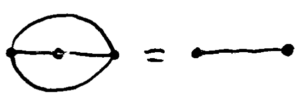
gives an isomorphism

$$A \xrightarrow{\sim} A^*$$
$$a \longmapsto g(a,-)$$

From this we get the "bubble move":

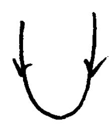


or dually:



3d

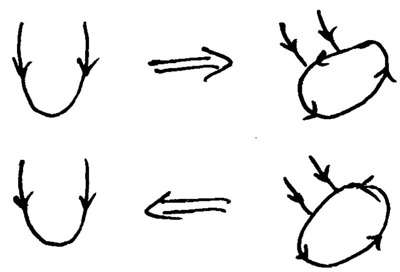
We need A to have a pairing



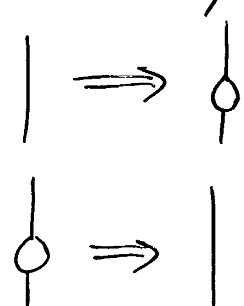
$$A \otimes A \xrightarrow{g} \mathrm{Vect}$$

(an exact functor)

together with 2-morphisms

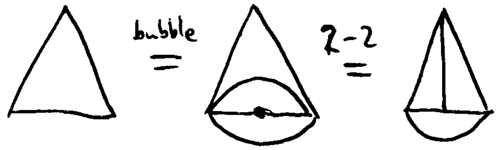


which need not be inverses! They will need to satisfy some coherence laws which imply the 1-4 Pachner move. To guess this, use duality to get 2-morphisms



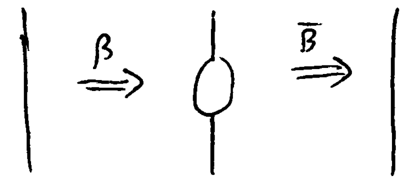
where now we assume g gives $A \simeq A^*$.

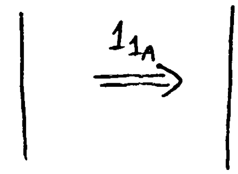
which implies the 1-3 move:



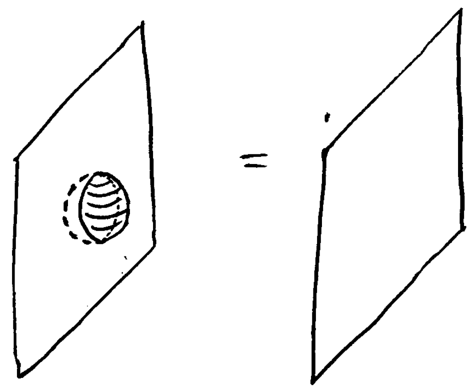
or dually:

We also want

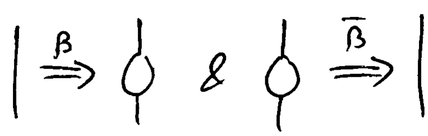


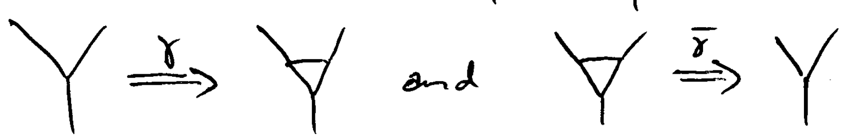


i.e., in spin foam notation:

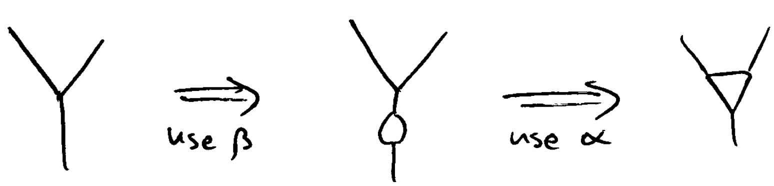


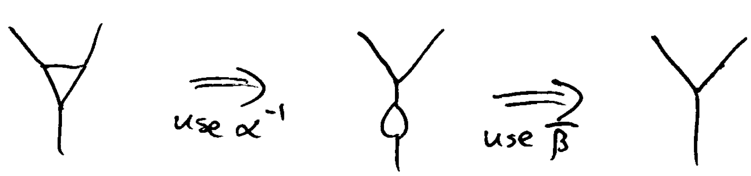
the 3d "bubble move"!

First, from  , let's get 2-morphisms

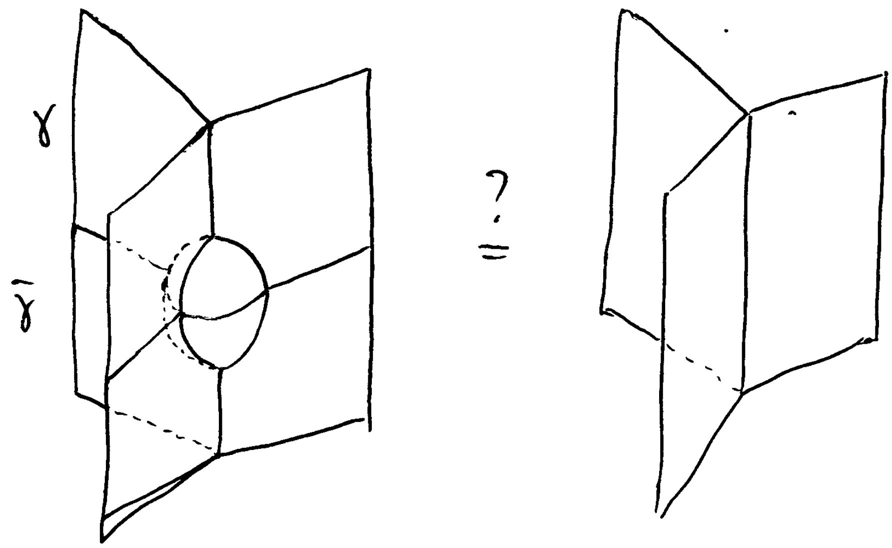


Here's how:

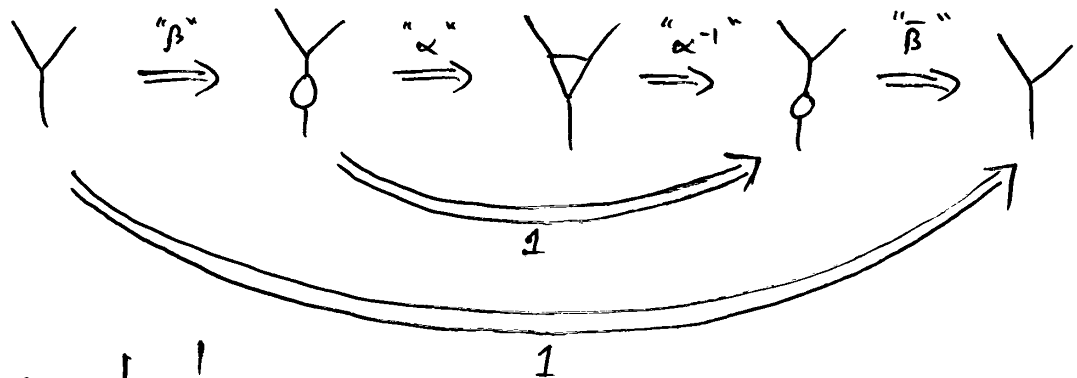




We know $\beta\bar{\beta} = 1$; can we show $\gamma\bar{\gamma} = 1$?



Yes! Here's how:



This commutes!