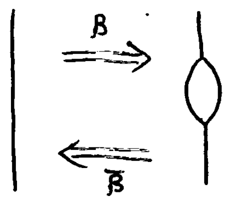
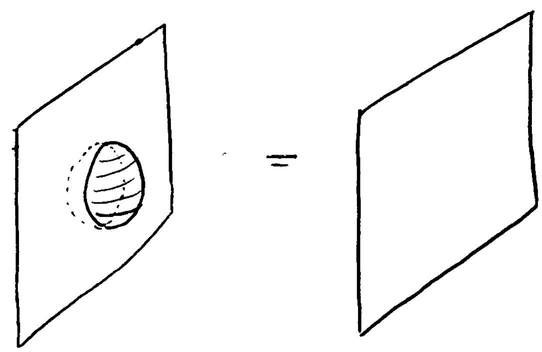
27 January 2005

Starting with a 2-algebra A equipped with a nondegenerate pairing $g: A \otimes A \to \mathrm{Vect}$, we then assumed the existence of 2-morphisms

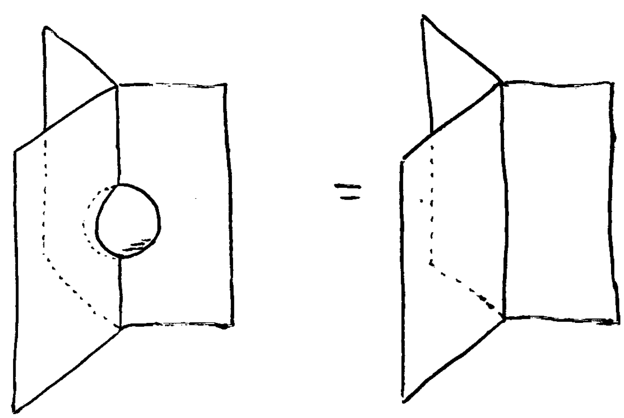


$$\begin{tikzcd} 1_A \arrow[r, bend left, Rightarrow, "\beta"] & m^\dagger m \arrow[l, bend left, Rightarrow, "\bar\beta"] \end{tikzcd}$$

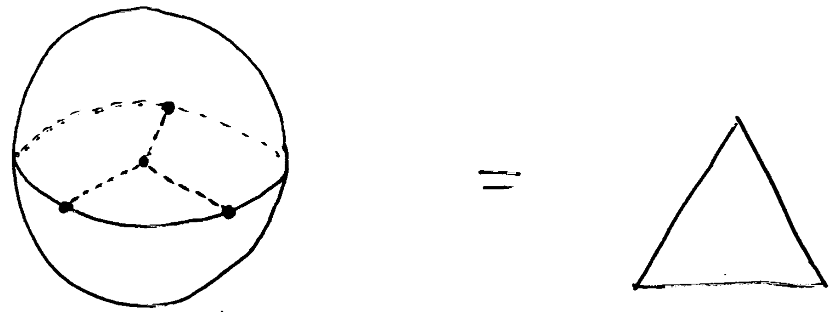
satisfying the "bubble move" equation $\beta\bar\beta = 1_{1_A}$:



From this we proved another version of the bubble move:

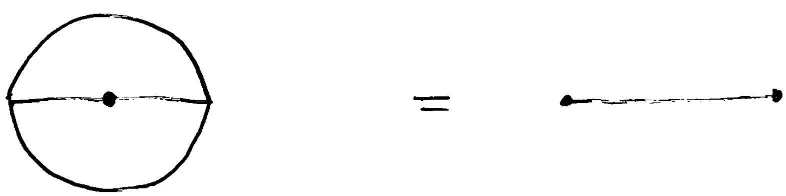


Now from this let's prove the 1-4 Pachner move, thus getting the full set of Pachner moves. Taking the Poincaré dual of the above "bubble move", we get...



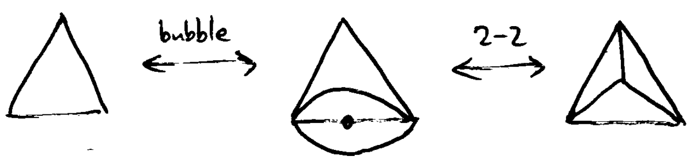
(2 tetrahedra sharing 3 triangles (i.e sharing all but one face each)

This is analogous to the 2d bubble move

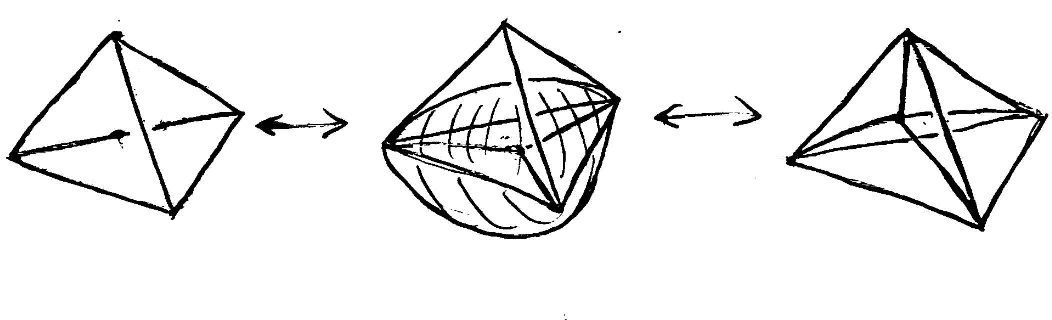


(2 triangles sharing all but one face each

So let's copy how we got the 1-3 move from the 2d bubble move:



to get the 1-4 move from the 3d bubble move

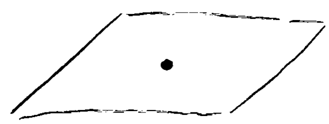


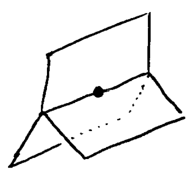
So, in principal we know how to get an extended 3d TQFT:

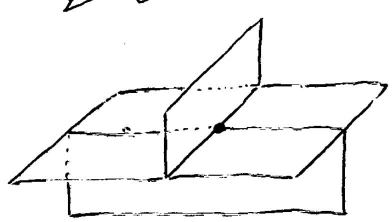
$$Z : 3\,\mathrm{Cob}_2 \longrightarrow 2\mathrm{Vect}$$

from a 2-algebra A with nondegenerate pairing & $\beta, \bar{\beta}$ s.t. $\beta\bar{\beta} = 1_{1_A}$.
Before doing examples like $A = \mathrm{Vect}[G]$ (categorified group algebra of a finite group), let's say a word about these surfaces we've been drawing!

A fake surface is a 2d CW complex where every point has a neighborhood that looks like either

1) 

2) 

3) 

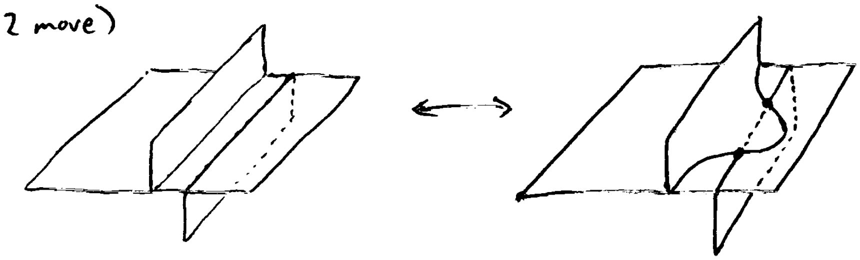
E.g. a generic bunch of soap suds!

Thm — Any topological 3-manifold can be made into a smooth or piecewise linear manifold, in an essentially unique way. (Not true in 4d!)

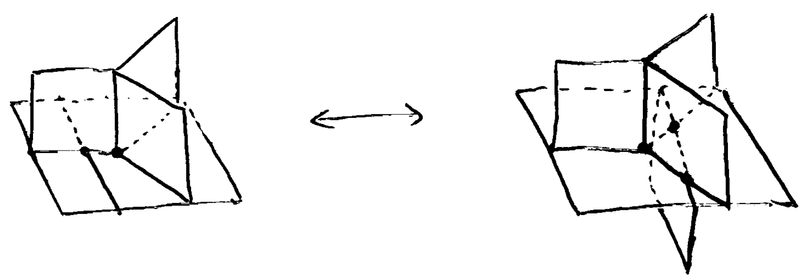
Thm (Matveev) — Suppose M is a compact 3-manifold. by previous thm., doesn't matter if we take smooth, PL, or Top.
Then we can embed a fake surface S in M such that $M-S$ is a disjoint union of open 3-balls (E.g. take S to be the dual 2-skeleton of a triangulation of M.)

You can recover M (up to isomorphism (diffeo. or PL-iso.)) from S. In fact, you can always choose S so that $M-S$ is <u>one</u> open 3-ball, & then S is called a <u>special spine</u> of M. Moreover, we can go between any two special spines of M using a finite sequence of homeomorphisms of M & <u>Matveev moves</u>

1) Lune move:
(or 0-2 move)



2) 2-3 Move



If we allow fake surfaces whose complement is a number of open balls, we need the bubble move as well:

The 2-3 move is the pentagon identity... what's the lune move?
It says the associator has an <u>inverse</u>!