1 February 2005

GAUGE THEORY

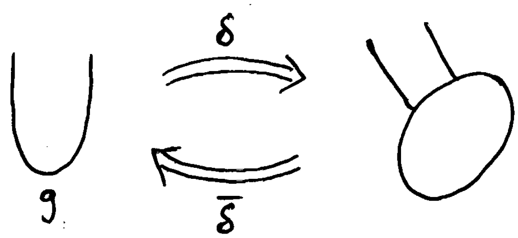
Let G be a finite group. Then we've shown that the group algebra $\mathbb{C}[G]$ is semisimple so it gives a 2d TQFT. We could also show that the "group 2-algebra" $\mathrm{Vect}[G]$ is a <u>semisimple</u> <u>2-algebra</u>: i.e. a 2-algebra A with a pairing

$$g : A \otimes A \longrightarrow \mathrm{Vect}$$

which is nondegenerate:

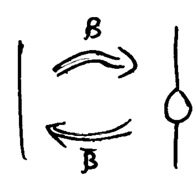
$$\begin{aligned} \#: A &\longrightarrow A^* \\ a &\longmapsto g(a,-) \end{aligned}$$

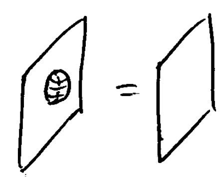
is an equivalence of 2-vector spaces, and there are 2-morphisms



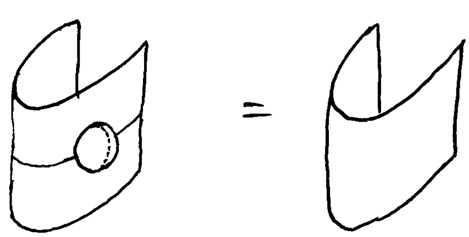
such that the bubble move equation holds:

$$\delta \bar{\delta} = 1_g$$

(This is equivalent to having 

giving )

We can draw $\delta\bar{\delta} = 1_g$ as

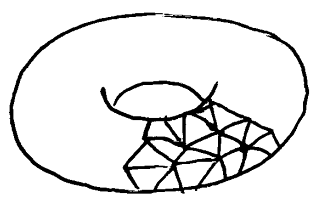


(Note: being semisimple is extra structure on a 2-algebra, namely δ & $\bar{\delta}$!)
(For Vect[G], I know one way to choose δ & $\bar{\delta}$; there may be more.)

Thus, Vect[G] gives a 3d extended TQFT. Of course, this pattern should continue: (n-1) Vect[G] is a semisimple n-algebra, & thus gives an (n+1)-dimensional extended TQFT. But, nobody has made sense of this and proved it.

But now, we'll tackle something different: what are these TQFTs like? We'll see that they're gauge theories: given a triangulated cobordism $M: S \to S'$, the operator $\tilde{Z}(M): \tilde{Z}(S) \to \tilde{Z}(S')$ can be computed as a "path integral" (actually a finite sum in this case) over connections on M. (Physicists call connections "gauge fields.") But, since M is a discrete structure (a triangulated manifold), we have to adapt the usual concept of connection to this context. Also, it's nontraditional to have the "gauge group" G be finite, but it simplifies things.

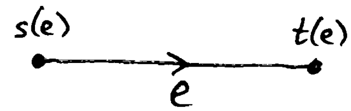
Suppose M is a triangulated manifold:



Let V be the set of vertices, E the set of edges, and arbitrarily choose a "direction" for each edge $e \in E$. We do this by choosing source and target maps

$$s, t : E \longrightarrow V$$

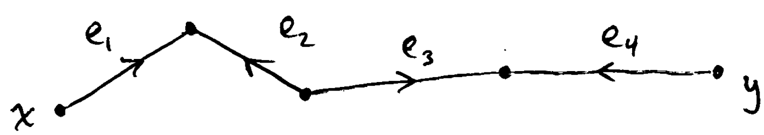
assigning to each edge its starting and finishing vertices.



Now we'll define connections & gauge transformations using only these data : $s, t : E \to V$ (a directed graph). A connection is just a map

$$A : E \longrightarrow G$$

saying how a point particle transforms as we move it along an edge $e \in E$, from $s(e)$ to $t(e)$. Given this, we can associate a group element to any "edge path"



Here's an edge path $\gamma = e_1 e_2^{-1} e_3 e_4^{-1}$ from x to y. Our connection assigns it an element

$$A(\gamma) = A(e_1) A(e_2)^{-1} A(e_3) A(e_4)^{-1} \in G$$

where we're extending the definition of A to edge paths.

(In fact, there's a groupoid P whose objects are vertices & whose morphisms are edge paths, & our connection becomes a functor

$$A : P \longrightarrow G$$

where the group G is seen as a 1-object groupoid:

$$A(1_v) = 1 \in G$$
$$A(\gamma\gamma') = A(\gamma)A(\gamma')$$

with 1_v the path that just sits at $v \in V$. These imply: $A(\gamma^{-1}) = A(\gamma)^{-1}$.)

The set of all connections will be called

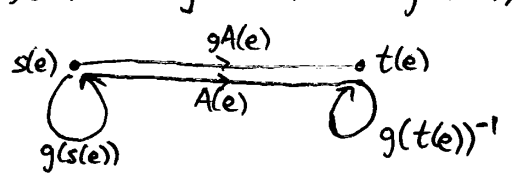
$$\mathcal{A} := G^E$$

Gauge transformations act on connections. A gauge transformation is a map

$$g : V \longrightarrow G$$

assigning a "change of viewpoint" transformation to each point in spacetime ($v \in V$). Given a gauge transformation g & a connection A, we get a new connection gA as follows

$$(gA)(e) = g(s(e))\,A(e)\,g(t(e))^{-1}$$



"Change viewpoint; move particle; change back."

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We're doing gauge theory on a graph:

$$\Gamma = \{ V \underset{t}{\overset{s}{\leftleftarrows}} E \}$$

& we define connections to be elements of

$$\mathcal{A} = \mathcal{A}(\Gamma) := G^E$$

where G is some fixed group, & gauge transformations to be elements of

$$\mathcal{G} = \mathcal{G}(\Gamma) = G^V$$

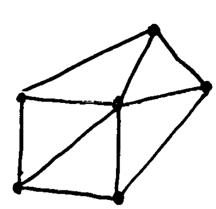
which form a group in the obvious way:

$$(gg')(v) = g(v)g'(v)$$

& which act on connections via

$$(gA)(e) = g(s(e))\, A(e)\, g(t(e))^{-1} \qquad g \in \mathcal{G},\ A \in \mathcal{A}.$$

Don't forget: our eventual goal is to express the partition function $\tilde{Z}(M)$ of a triangulated manifold M as an integral over $\mathcal{A}(\Gamma)$, where Γ is the 1-skeleton of M.



In fact this integral can be done over just certain special "flat" connections, which form a subset $\mathcal{A}_0 \subseteq \mathcal{A}$.

Let's make a sophisticated digression. Last time we saw that connections could be thought of as functors $A : P(\Gamma) \to G$ where $P(\Gamma)$ is the "path groupoid" of Γ, where

- objects are vertices $v \in V$
- morphisms are expressions like

$$e_1 e_2^{-1} e_3 e_4 \qquad \begin{tikzcd} \bullet \arrow[r, "1"'] & \bullet & \bullet \arrow[l, "2"] \arrow[r, "3"'] & \bullet \arrow[r, "4"'] & \bullet \end{tikzcd}$$

where the source/target of each e_i matches those of e_{i+1} in a hopefully obvious way, modulo relations coming from $ee^{-1} = 1$, $e^{-1}e = 1$

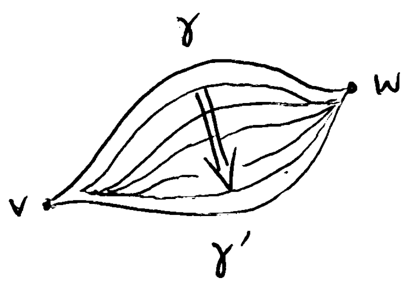
From this point of view, gauge transformations give natural transformations: $g \in \mathcal{G}$ & $A \in \mathcal{A}$ give a new connection gA & a natural transformation

$$\begin{tikzcd} P(\Gamma) \arrow[r, bend left=50, "A"{name=U}] \arrow[r, bend right=50, "gA"'{name=D}] & G \arrow[Rightarrow, from=U, to=D, "{(g,A)}"] \end{tikzcd}$$

which assigns to any object $v \in P(\Gamma)$ a morphism $g(v) \in G$. Naturality says that this square commutes:

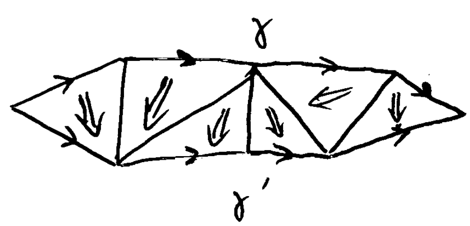
$$
\begin{tikzcd}
A(v) \arrow[r, "A(\gamma)"] \arrow[d, "g(v)"'] & A(w) \arrow[d, "g(w)"] \\
(gA)(v) \arrow[r, "(gA)(\gamma)"] & (gA)(w)
\end{tikzcd}
$$

Back to flat connections: Roughly, a connection is "flat" if it assigns the same group element to homotopic paths:



(where endpoints remain fixed)

This makes sense for paths in a manifold, but not in a mere graph. In fact, we only need to know the triangles in our triangulated manifold to define a concept of "simplicial homotopy":



where we "slide γ over triangles to get γ'." So, let's define a concept of <u>simplicial</u> 2-graph

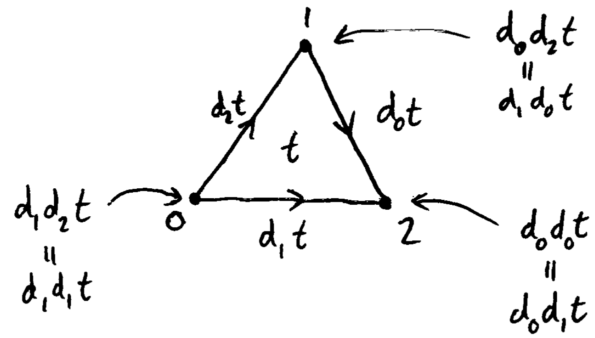
A <u>simplicial 2-graph</u> will be:

- a set V of vertices
- a set E of edges
- a set T of triangles

with maps

$$V \underset{d_1}{\overset{d_0}{\leftleftarrows}} E \;\overset{d_0,\, d_1,\, d_2}{\leftarrow\!\!\!\leftarrow\!\!\!\leftarrow}\; T$$

where d_i means "leave out the ith vertex":

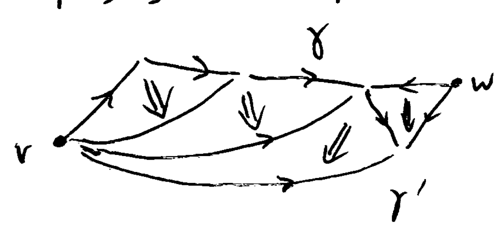


satisfying

$$d_1 d_1 t = d_1 d_2 t$$
$$d_0 d_2 t = d_1 d_0 t$$
$$d_0 d_0 t = d_0 d_1 t.$$

Associated to any simplicial 2-graph there's a 2-groupoid where:

- objects are vertices :
- morphisms are edge paths :  $\gamma : v \to w$
- 2-morphisms are simplicial homotopies between edge paths

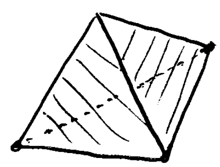


Given a simplicial 2-graph H, it has an underlying graph Γ:

$$V \overset{d_0 = t}{\underset{d_1 = s}{\leftleftarrows}} E$$

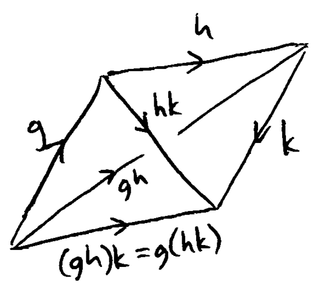
so we can define connections & gauge transformations on H to be those on Γ. But we can also define a conn. to be flat if $A(\gamma) = A(\gamma')$ whenever there's a simplicial homotopy $f: \gamma \Rightarrow \gamma'$.

For example take $G = \mathbb{Z}/2$ and let

$H =$ 

What are all the flat connections on H? Or: how many?

There are $|G^E| = |G|^{|E|} = 2^6$ connections, but most aren't flat



We can pick $g, h, k \in \mathbb{Z}/2$ arbitrarily and flatness determines the rest — so there are $2^3 = 8$ flat connections. (For an n-simplex, there are $|G|^n$ flat conns!)