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Now let's prove this:

<u>Thm</u> – Suppose G is a group & M is a connected triangulated manifold with chosen vertex $*$. Let $\pi_1(M)$ be the fundamental group of M & let $\mathcal{A}_0(M)$ be the set of flat G-connections on M & let $\mathcal{G}_0(M)$ be the subgroup of the group $\mathcal{G}(M)$ of gauge transformations consisting of gauge transformations g with $g(*) = 1$. Then:

$$\mathcal{A}_0(M)/\mathcal{G}_0(M) \cong \hom(\pi_1(M), G)$$

<u>Proof</u> – Recall that for any triangulated manifold M there's the <u>path groupoid</u> PM with

objects = vertices of M
morphisms = edge paths in M $\qquad (ee^{-1} = 1, e^{-1}e = 1)$

If M is connected, there's just one isomorphism class of objects in PM, so we can get an equivalent groupoid (a skeleton of PM) by taking one object $* \in PM$ & forming the <u>loop group</u> ΩM with

object = $*$
morphisms = edge paths $\gamma : * \to *$
i.e. "edge loops based at $*$"

We have an equivalence of categories

$$\Omega M \hookrightarrow PM$$

Next; recall that for any triangulated manifold M there's a <u>fundamental groupoid</u> $\Pi_1 M$ with

objects = vertices of M
morphisms = simplicial homotopy classes of edge paths in M

Again, when M is connected we can pick any object $* \in \Pi_1 M$ and define a skeleton of $\Pi_1 M$, the <u>fundamental group</u> $\pi_1 M$ with

object = $*$
morphisms = simplicial homotopy classes of loops $\gamma : * \to *$.

We have an equivalence:

$$\pi_1 M \hookrightarrow \Pi_1 M$$

Note we have a commutative square:

$$\begin{tikzcd}
\Omega M \arrow[r, hook] \arrow[d] & PM \arrow[d] \\
\pi_1 M \arrow[r, hook] & \Pi_1 M
\end{tikzcd}$$

where the downward arrows are quotient maps (essentially surjective and full). This square gives another square

$$\begin{tikzcd}
\hom(\Omega M, G) & \hom(PM, G) \arrow[l] \\
\hom(\pi_1 M, G) \arrow[u] & \hom(\Pi_1 M, G) \arrow[l] \arrow[u]
\end{tikzcd}$$

Note: if X & Y are categories, $\hom(X,Y)$ is a <u>category</u> where

$$\text{objects} = \text{functors } f: X \to Y$$
$$\text{morphisms} = \text{natural transformations}$$

((<u>Cat</u> is a <u>closed</u> category))

If X & Y are groupoids this $\hom(X,Y)$ is a groupoid.

Note:

$\hom(PM, G)$ has as $\begin{cases} \text{objects: } G\text{-connections on } M : A \in \mathcal{A}(M) \\ \text{morphisms: gauge transformations} \\ \qquad g: A \longmapsto A' \text{ with } g \in \mathcal{G}(M) \end{cases}$

$\hom(\Pi_1 M, G)$ has as $\begin{cases} \text{objects: } \underline{\text{flat}}\ G\text{-connections on } M : A \in \mathcal{A}_0(M) \\ \text{morphisms: gauge transformations} \\ \qquad g: A \longmapsto A' \text{ with } g \in \mathcal{G}(M) \end{cases}$

So

$$\mathcal{A}_0(M)/\mathcal{G}(M) = \text{isomorphism classes of objects in } \hom(\Pi_1 M, G)$$

and

$$\mathcal{A}_0(M)/\mathcal{G}_0(M) = \text{objects in } \hom(\Pi_1 M, G) \text{ mod isomorphisms } g \text{ with } g(*) = 1$$

Similarly:

$\hom(\pi_1 M, G)$ has as $\begin{cases} \text{objects: homomorphisms } f: \pi_1 M \to G \\ \text{morphisms: elts } g \in G \text{ with } f'(x) = g f(x) g^{-1} \end{cases}$

i.e.
$$\begin{tikzcd} * \arrow[r, "f(x)"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & * \arrow[d, "g"] \\ * \arrow[r, "f'(x)"'] & * \end{tikzcd}$$
$x \in \pi_1 M$

To show that $\mathcal{A}_0(M)/\mathcal{G}_0(M)$ is $\cong$ the set of homomorphisms $f: \pi_1 M \to G$ .... note: {objects in $\hom(\pi_1 M, G)$ mod isos g with $g(*) = 1$} = {homomorphisms $f: \pi_1 M \to G$} while {objects in $\hom(\Pi_1 M, G)$ mod isos g w. $g(*) = 1$} = $\mathcal{A}_0(M)/\mathcal{G}_0(M)$ & these sets are isomorphic because $\pi_1 M$ & $\Pi_1 M$ are equivalent (and we are modding out by the same natural isos. in both $\hom(\Pi_1 M, G)$ and $\hom(\pi_1 M, G)$) ▨

We've shown much more, e.g.: for any triangulated manifold M,

$$\mathcal{A}(M)/\mathcal{G}(M) = \{\text{nat. iso. classes of functors } A: PM \to G\}$$

$$\mathcal{A}_0(M)/\mathcal{G}(M) = \{\text{nat. iso. classes of functors } A: \Pi_1 M \to G\}$$

and if M is <u>connected</u>, then

$$\mathcal{A}(M)/\mathcal{G}(M) = \{\text{conjugacy classes of homomorphisms } f: \Omega M \to G\}$$

(where $f, f': \Omega M \to G$ are <u>conjugate</u> if $f'(x) = g f(x) g^{-1}$ $\exists g \in G$

$$\mathcal{A}_0(M)/\mathcal{G}(M) = \{\text{conjugacy classes of homomorphisms } f: \pi_1 M \to G\}$$

Vast amounts of energy have been spent studying this last space for various choices of M & G – it's called the "moduli space of flat connections"

Recap:

For any finite group G we get a 2D TQFT Z from the group algebra $\mathbb{C}[G]$, and if M is a compact oriented connected 2-manifold,

$$Z(M) = \sum_{\mathcal{A}_0(M)/\mathcal{G}_0(M)} \frac{1}{|G|^{2g-1}}$$

where g is the genus of M, i.e. it's a "path integral" over the space $\mathcal{A}_0(M)/\mathcal{G}_0(M)$ of flat connections mod gauge transformations that equal 1 at some chosen $* \in M$. The genus g is related to the Euler characteristic $\chi(M)$ by

$$\chi(M) = 2 - 2g$$

so

$$2g = 2 - \chi(M)$$
$$2g - 1 = 1 - \chi(M)$$

&

$$Z(M) = \sum_{\mathcal{A}_0(M)/\mathcal{G}_0(M)} |G|^{\chi(M)-1}$$

The annoying $|G|^{-1}$ comes from $\mathcal{G}(M)/\mathcal{G}_0(M) \cong G$ & indeed we would have

$$Z(M) = \sum_{\mathcal{A}_0(M)/\mathcal{G}(M)}$$

But this isn't true even though $\mathcal{A}_0(M)/\mathcal{G}(M) \cong \dfrac{\mathcal{A}_0(M)/\mathcal{G}_0(M)}{G}$, since it's not true that

E.g. $|3/\mathbb{Z}_2| \neq \frac{|3|}{2}$

$$|\mathcal{A}_0(M)/\mathcal{G}(M)| = \frac{|\mathcal{A}_0(M)/\mathcal{G}_0(M)|}{|G|}$$

unless G acts freely on $\mathcal{A}_0(M)/\mathcal{G}(M)$.
(suggests we really should be using groupoid cardinality and the weak quotient)

If G acts freely on $\mathcal{A}_0(M)/\mathcal{G}_0(M)$, then the formula holds and

$$Z(M) = |\mathcal{A}_0(M)/\mathcal{G}(M)| \cdot |G|^{\chi(M)}$$

In general, we only have

$$Z(M) = \frac{|\mathcal{A}_0(M)/\mathcal{G}_0(M)|}{|G|}$$

Or (see last year's notes) we could form the weak quotient $\mathcal{A}_0(M)//\mathcal{G}(M)$ – a <u>groupoid</u>, & use groupoid cardinality to get:

$$Z(M) = |\mathcal{A}_0(M)//\mathcal{G}(M)|\,|G|^{\chi(M)}.$$

$\mathcal{A}_0(M)//\mathcal{G}(M)$ is called the <u>moduli stack</u> of flat connections, and keeps track of how certain flat connections get mapped to themselves by gauge transformations at the special vertex $*$. These are called <u>reducible</u> connections

$|G|^{\chi(M)}$ resembles e^{-S} & indeed the action in 2d gravity is (proportional to) $\chi(M)$. In any dimension, there's an "Euler TQFT" with $Z(M) = \alpha^{\chi(M)}$ for any closed manifold M (and any fixed $\alpha \in \mathbb{C}$).

All this stuff works in any dimension. In dimension 3, we get a TQFT Z from the group 2-algebra $\mathrm{Vect}[G]$, which has a basis of objects $\mathbb{C}_g$ $(g \in G)$ with

$$\mathbb{C}_g \otimes \mathbb{C}_h = \mathbb{C}_{gh}$$

& this was studied by Dijkgraaf & Witten (the "Dijkgraaf - Witten model") & later by Dan Freed & Frank Quinn. They saw if M is a compact oriented 3-manifold,

$$Z(M) = |A_0(M) /\!/ \mathcal{G}(M)|$$

What happened to the $|G|^{\chi(M)}$? Answer: $\chi(M) = 0$ for every compact oriented 3-manifold. In any dimension n

$$\chi(M) = \sum_{i=1}^{n} (-1)^i \dim |H_i(M,\mathbb{R})|$$

$$\& \quad H_i(M,\mathbb{R}) \cong H^*_{n-i}(M,\mathbb{R})$$

Poincaré duality

so we get 0 when n is odd:

$$\dim H_0 - \dim H_1 + \dim H_2 - \dim H_3$$

So in fact $|G|^{\chi(M)}$ is really there; it's just hiding.

In fact, in <u>any</u> dimension we get a TQFT with

$$Z(M) = |A_0(M)/\!\!/\mathcal{G}(M)|\ |G|^{\chi(M)}$$

but people haven't constructed them using $2\mathrm{Vect}[G]$, $3\mathrm{Vect}[G]$, $4\mathrm{Vect}[G]$, ... etc., except in dimensions ≤ 4. (Marco Mackaay has built the 4d TQFT using $2\mathrm{Vect}[G]$.)

<u>Twisting</u>:

Dijkgraaf & Witten showed how to modify or "twist" the TQFTs we've been discussing, using group cohomology. For example, in 2 dimensions, let's take $\mathbb{C}[G]$ with product:

$$\delta_g \delta_h = \delta_{gh}$$

where

$$\delta_g(h) = \begin{cases} 0 & g \neq h \\ 1 & g = h \end{cases}$$

and then define a new "<u>twisted</u>" <u>product</u>, $*$, as follows...

$$\delta_g * \delta_h = c(g,h)\,\delta_{gh}$$

where
$$c : G^2 \longrightarrow \mathbb{C} - \{0\}.$$

To get a TQFT we want this to yield a new semisimple algebra — in particular, $*$ had better be associative. Let's see when this is true:

$$\begin{array}{ccc}
(\delta_g * \delta_h) * \delta_k & \overset{\text{want}}{=\!=} & \delta_g * (\delta_h * \delta_k) \\
\| & & \| \\
c(g,h)\,\delta_{gh} * \delta_k & & c(h,k)\,\delta_g * \delta_{hk} \\
\| & & \| \\
c(g,h)\,c(gh,k)\,\delta_{ghk} & & c(h,k)\,c(g,hk)\,\delta_{ghk}
\end{array}$$

so we need c to be a 2-cocycle, i.e.

$$c(g,h)\,c(gh,k) = c(h,k)\,c(g,hk)$$

or better:

$$\underbrace{c(h,k)}_{\text{omit first one}}\;\underbrace{c(gh,k)^{-1}}_{\text{multiply first pair}}\;\underbrace{c(g,hk)}_{\text{multiply second pair}}\;\underbrace{c(g,h)^{-1}}_{\text{omit last one}} = 1$$

To keep δ_1 being the multiplicative identity, we also need

$$\delta_1 * \delta_g = c(1,g)\,\delta_g = \delta_g$$
$$\delta_g * \delta_1 = c(g,1)\,\delta_g = \delta_g$$

i.e.:

$$c(1,g) = c(g,1) = 1$$

i.e. c is a <u>normalized</u> cocycle.

Next time, we'll twist the <u>associator</u> in $\mathrm{Vect}[G]$ using a normalized 3-cocycle.