Twisting by a Cocycle, continued...

We saw that we could "twist" the multiplication in the group algebra $\mathbb{C}[G]$ as follows:

$$\underbrace{\delta_g * \delta_h}_{\text{new multiplication}} = c(g,h)\, \underbrace{\delta_g \delta_h}_{\substack{\text{old multiplication}\\ (=\delta_{gh})}}$$

where $c: G^2 \longrightarrow \mathbb{C} - \{0\}$ and $*$ is associative if c is a <u>2-cocycle</u>:

$$\begin{array}{ccc} (\delta_g * \delta_h) * \delta_k & = & \delta_g * (\delta_h * \delta_k) \\ \| & & \| \\ c(g,h)\, \delta_{gh} * \delta_k & & c(h,k)\, \delta_g \delta_{hk} \\ \| & & \| \\ c(g,h) c(gh,k)\, \delta_{ghk} & & c(h,k) c(g,hk)\, \delta_{ghk} \end{array}$$

if

$$c(g,h) c(gh,k) = c(h,k) c(g,hk)$$

or

$$c(h,k) c(gh,k)^{-1} c(g,hk) c(g,h)^{-1} = 1$$

Also, δ_1 is the identity for $*$ if c is <u>normalized</u>:

$$c(1,g) = c(g,1) = 1 \qquad \forall g \in G$$

If $c(g,h)$ is also sufficiently close to 1, our new "twisted" algebra $(\mathbb{C}[G], *, 1)$ is still semisimple, because we've only changed the killing form $tr(L_v L_w)$ a little bit, so it remains nondegenerate! So in this case we get a 2d TQFT, a slightly

"perturbed" version of the TQFT coming from $(\mathbb{C}[G], \cdot, 1)$.
So, perturbing TQFTs is related to cohomology!

Now let's categorify this idea: take the 2-algebre $\mathrm{Vect}[G]$ and twist it! We'll keep the same tensor product, namely

$\mathbb{C}_g$ = v.s. valued function on the group analogous to δ-function $\mathbb{C}$ at g and 0 elsewhere.

$$\mathbb{C}_g \otimes \mathbb{C}_h = \mathbb{C}_{gh}$$

(convolution tensor product)

(because this is hard to twist in an interesting — though it might be possible) But we'll twist the associator! The associator in $\mathrm{Vect}[G]$ is the obvious isomorphism

$$a_{g,h,k} : (\mathbb{C}_g \otimes \mathbb{C}_h) \otimes \mathbb{C}_k \xrightarrow{\sim} \mathbb{C}_g \otimes (\mathbb{C}_h \otimes \mathbb{C}_k)$$

gives $(\mathbb{C} \otimes \mathbb{C}) \otimes \mathbb{C} \longrightarrow \mathbb{C} \otimes (\mathbb{C} \otimes \mathbb{C})$ at ghk at ghk

& we can define a new associator

$$\alpha_{g,h,k} := c(g,h,k)\, a_{g,h,k}$$

where we use the fact that hom-sets in a 2-vector space are actually vector spaces, so we can multiply the morphism $a_{g,h,k}$ by a number $c(g,h,k)$! But this new associator will only satisfy the pentagon equation if

$$c : G^3 \longrightarrow \mathbb{C} - \{0\}$$

satisfies some equation, in which case we call it a <u>3-cocycle</u>:

$$
\begin{tikzcd}[column sep=small]
 & ((\mathbb{C}_g \otimes \mathbb{C}_h) \otimes \mathbb{C}_k) \otimes \mathbb{C}_\ell \arrow[dl, "\alpha_{gh,k,\ell}"'] \arrow[dr, "\alpha_{g,h,k} \otimes \mathbb{C}_\ell"] & \\
(\mathbb{C}_g \otimes \mathbb{C}_h) \otimes (\mathbb{C}_k \otimes \mathbb{C}_\ell) \arrow[ddr, "\alpha_{g,h,k\ell}"] & & (\mathbb{C}_g \otimes (\mathbb{C}_h \otimes \mathbb{C}_k)) \otimes \mathbb{C}_\ell \arrow[d, "\alpha_{g,hk,\ell}"] \\
 & & \mathbb{C}_g \otimes ((\mathbb{C}_h \otimes \mathbb{C}_k) \otimes \mathbb{C}_\ell) \arrow[dl, "\mathbb{C}_g \otimes \alpha_{h,k,\ell}"] \\
 & \mathbb{C}_g \otimes (\mathbb{C}_h \otimes (\mathbb{C}_k \otimes \mathbb{C}_\ell)) &
\end{tikzcd}
$$

will commute iff

$$c(gh,k,\ell)\,c(g,h,k\ell) = c(g,h,k)\,c(g,hk,\ell)\,c(h,k,\ell)$$

since the corresponding diagram with a instead of α commutes.
We can rewrite this condition as

$$c(h,k,\ell)\,c(gh,k,\ell)^{-1}\,c(g,hk,\ell)\,c(g,h,k\ell)^{-1}\,c(g,h,k) = 1 .$$

We see a similar pattern as in our 2-cocycle before!
But to get a full fledged 2-algebra with the <u>same</u> unit object $\mathbb{C}_1 \in \mathrm{Vect}[G]$ & <u>same</u> left & right unit laws:

$$\ell_g : \mathbb{C}_1 \otimes \mathbb{C}_g \xrightarrow{\sim} \mathbb{C}_g$$
$$r_g : \mathbb{C}_g \otimes \mathbb{C}_1 \xrightarrow{\sim} \mathbb{C}_g$$

as in $\mathrm{Vect}[G]$, we need the triangle

$$\begin{tikzcd}
(\mathbb{C}_g \otimes \mathbb{C}_1) \otimes \mathbb{C}_h \arrow[rr, "\alpha_{g,1,h}"] \arrow[dr, "r_g \otimes \mathbb{C}_h"'] & & \mathbb{C}_g \otimes (\mathbb{C}_1 \otimes \mathbb{C}_h) \arrow[dl, "\mathbb{C}_g \otimes \ell_h"] \\
& \mathbb{C}_g \otimes \mathbb{C}_h &
\end{tikzcd}$$

to commute. Since it commuted with $a_{g,1,h}$ in place of $\alpha_{g,1,h}$, we need

$$c(g,1,h) = 1 \quad \forall g,h \in G$$

In fact, this implies

$$c(1,g,h) = 1 \quad \& \quad c(g,h,1) = 1$$

as well (using the 3-cocycle condition). You can also see this from the commuting of

$$\begin{tikzcd}
(\mathbb{C}_1 \otimes \mathbb{C}_g) \otimes \mathbb{C}_h \arrow[rr, "\alpha_{1,g,h}"] \arrow[dr, "\ell_g \otimes \mathbb{C}_h"'] & & \mathbb{C}_1 \otimes (\mathbb{C}_g \otimes \mathbb{C}_h) \arrow[dl, "\ell_{gh}"] \\
& \mathbb{C}_g \otimes \mathbb{C}_h &
\end{tikzcd}$$

(which also commutes automatically by MacLane's coherence thm.) and some other diagram.

If these conditions hold:

$$c(1,g,h) = c(g,1,h) = c(g,h,1) = 1$$

we say c is <u>normalized</u> and Vect$[G]$ becomes a 2-algebra with the twisted associator α.

But: how do we find 2- & 3-cocycles?
And: where do the cocycle conditions really come from, and how do we define n-cocycles $\forall n$?

We saw:

2-cocycle equation $\cong$ associativity $\leftarrow$ (a law for monoids)
3-cocycle equation $\cong$ pentagon identity $\leftarrow$ (a law for monoidal categories)

so we can guess that the n-cocycle equation is related to a law holding in monoidal $(n-2)$-categories. This is true! But this is not an easy way to guess the n-cocycle condition, & historically the n-cocycle condition came first. This condition was first deeply understood by Eilenberg & Maclane, as follows:

$$
\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[dr, phantom, "c(g{,}h)", pos=0.3] & \bullet \arrow[d, "k"] \\
\bullet \arrow[u, "g"] \arrow[ur, "gh"] \arrow[r, "(gh)k"'] \arrow[ur, phantom, "c(gh{,}k)", pos=0.7, shift right=4ex] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[dr, "hk"] & \bullet \arrow[d, "k"] \arrow[dl, phantom, "c(h{,}k)", pos=0.3] \\
\bullet \arrow[u, "g"] \arrow[r, "g(hk)"'] \arrow[ur, phantom, "c(g{,}hk)", pos=0.3] & \bullet
\end{tikzcd}
$$

Twist the product in $\mathbb{C}[G]$ by c, and demand associativity.

See? Associativity is all about tetrahedra (= 3-simplices), & the 4 terms in the 2-cocycle equation come from the 4 faces of a 3-simplex. Likewise, the pentagon eq. is all about 4-simplices, & the 5 terms in the 3-cocycle condition come from the 5 faces in the 4-simplex.

And so on in higher dimensions...

Twisting by a Cocycle, continued...

We saw last time that to "twist" a TQFT based on the finite group G, we can use an "n-cocycle": a function

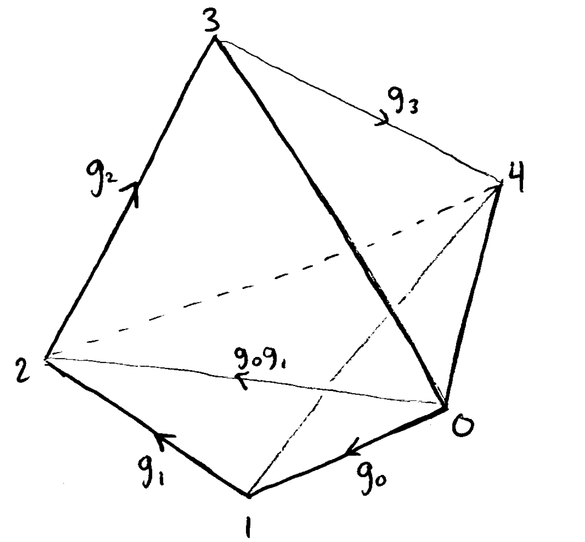
$$c: G^n \longrightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$$

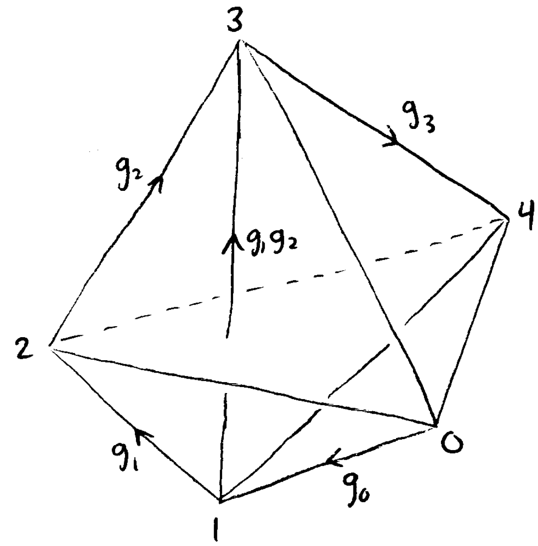
multiplicative group of nonzero complex numbers

satisfying a certain equation which we worked out for $n=2,3$. We began to see that this equation arises from pondering the $(n+1)$-simplex. We did this for $n=2$, but we'll see the pattern if we consider $n=3$. The 3-cocycle equation:

$$c(g_1,g_2,g_3)\,c(g_0g_1,g_2,g_3)^{-1}\,c(g_0,g_1g_2,g_3)\,c(g_0,g_1,g_2g_3)^{-1}\,c(g_0,g_1,g_2)=1$$

comes from the pentagon identity for the associator $\alpha_{g,h,k} = c(g,h,k)\,a_{g,h,e}$. But the pentagon identity is secretly the 2-3 Pachner move going from the "front" to the "back" of a 4-simplex, so let's understand the above equation using this:





2 tetrahedra with vertices:		3 tetrahedra with vertices:	
0234	$c(g_0g_1, g_2, g_3)$	1234	$c(g_1, g_2, g_3)$
0124	$c(g_0, g_1, g_2g_3)$	0134	$c(g_0, g_1g_2, g_3)$
		0123	$c(g_0, g_1, g_2)$

We specify (tetrahedral) faces of the 4-simplex by leaving out one vertex. The ones ones on the left leave out an odd vertex; those on the right leave out an even vertex. The 3-cocycle condition says the product of the 2 c's on the left equals the product of the 3 c's on the right!

In general, an n-simplex has $n+1$ vertices which we can label $0, 1, \ldots, n$. Assigning a group element g_i to the edge $i(i+1)$ determines a flat connection on the simplex where the edge ij $(i<j)$ gets the group element $g_i g_{i+1} \cdots g_{j-1}$. Given $c : G^n \to \mathbb{C}^*$, each $(n-1)$-dimensional face of the n-simplex gets assigned a number: the face $0123 \cdots \hat{i} \cdots n$ gets the number

$$c(g_0, g_1, \ldots, g_{i-1}g_i, \ldots, g_n)$$

unless $i=0$ or n, which give

$$c(g_1, \ldots, g_n) \text{ and } c(g_0, \ldots, g_{n-1}) \text{ respectively.}$$

Generalizing from $\mathbb{C}^*$ to any abelian group A (with operation $+$), we get:

<u>Def:</u> Given a group G & an abelian group A, an n-cochain on G valued in A is a map

$$c : G^n \longrightarrow A$$

Given an n-cochain c, we define its <u>coboundary</u> dc to be this $(n+1)$-cochain:

$$\begin{aligned} dc(g_0, \ldots, g_n) = {} & c(g_1, g_2, \ldots, g_n) - c(g_0 g_1, g_2, \ldots, g_n) \\ & + c(g_0, g_1 g_2, \ldots, g_n) - \cdots + (-1)^n c(g_0, \ldots, g_{n-1} g_n) \\ & + (-1)^{n+1} c(g_0, \ldots, g_{n-1}) \end{aligned}$$

We say c is an <u>n-cocycle</u> if $dc = 0$, & an <u>n-coboundary</u> if $c = dx$ for some $(n-1)$-cochain x.

<u>Thm:</u> $ddc = 0$, i.e. every coboundary is a cocycle.

<u>Proof:</u> Tiresome calculation or blinding flash of insight. ■

In this terminology, we've seen:

We can twist the group algebra $\mathbb{C}[G]$ by any 2-cocycle

$$c : G^2 \longrightarrow \mathbb{C}^*$$

to get an algebra, which is semisimple & thus gives a 2d TQFT when c is "sufficiently small" i.e. $|c-1| < \delta$ for some $\delta > 0$.

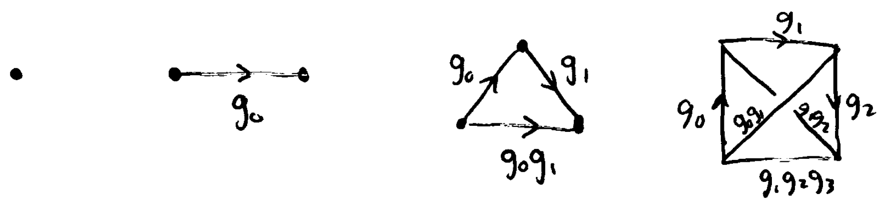
In general we should guess that we can twist the "group $(n-1)$-algebra" $(n-2)\mathrm{Vect}[G]$ by any n-cocycle $c : G^n \to \mathbb{C}^*$, which gives an n-dim TQFT when c is small. The main evidence is a certain <u>other</u> method for creating n-dim TQFTs from n-cocycles, which should give the same result but avoids the n-category theory lurking behind such as-yet-undefined concepts as "$(n-1)$-algebra", and "$(n-2)$-vector space".

Note: we've seen that the n-cocycle equation "is" the $\left(\lfloor \frac{n+2}{2} \rfloor, \lceil \frac{n+2}{2} \rceil\right)$ Pachner move, e.g. the $(2,3)$ Pachner move if $n = 3$!

Next:

Group cohomology is topology in disguise!

A hint: these pictures we've been drawing lately:



...

are views of a space associated to the group G, called its <u>classifying space</u>.