1 March 2005

Group Cohomology & TQFTs : Summary (with a boring twist)

At least for $n = 2, 3$ we've seen how to "twist" the n-dim TQFT that we get from $(n-1)\mathrm{Vect}[G]$ (e.g.: $\mathbb{C}[G]$ if $n=2$, $\mathrm{Vect}[G]$ if $n=3$) using an <u>n-cocycle</u>

$$c : G^n \longrightarrow \mathbb{C}^*$$

with

$$dc = 0.$$

But what about <u>n-coboundaries</u>, i.e.

$$c : G^n \longrightarrow \mathbb{C}^*$$

with

$$c = dx$$

for some

$$x : G^{n-1} \longrightarrow \mathbb{C}^* \quad ?$$

These are specially boring cocycles that give "trivial" ways to twist $(n-1)\mathrm{Vect}[G]$, i.e. ways that give an equivalent $(n-1)$-algebra & thus an isomorphic TQFT.

<u>Example:</u> $n = 2$.

We can take the group algebra $\mathbb{C}[G]$ and instead of using the standard basis $\{\delta_g\}$ let's pick another basis:

$$\varepsilon_g = x(g)\,\delta_g$$

where

$$x : G \longrightarrow \mathbb{C}^*$$

is any function.

Now we have

$$\begin{aligned}\varepsilon_g \varepsilon_h &= x(g)x(h)\,\delta_g\,\delta_h \\ &= x(g)x(h)\,\delta_{gh} \\ &= x(g)\,x(h)\,x(gh)^{-1}\,\varepsilon_{gh}\end{aligned}$$

– same algebra described using the new basis.

So: if we take $\mathbb{C}[G]$ and twist it, defining

$$\delta_g * \delta_h = c(g,h)\,\delta_{gh}$$

where

$$c(g,h) = x(g)x(h)\,x(gh)^{-1}$$

we must get an algebra isomorphic to $\mathbb{C}[G]$. So this $c: G^2 \longrightarrow \mathbb{C}^*$ gives a "trivial" way to twist $\mathbb{C}[G]$. But this formula for c in terms of x is just "$dx = c$" written in multiplicative form:

$$dx(g,h) = x(h)\,x(gh)^{-1}\,x(g)$$

So: 2-coboundaries $c: G^2 \longrightarrow \mathbb{C}^*$ are precisely these "trivial" ways to twist $\mathbb{C}[G]$!

Similarly, in 3d, if we twist the associator in $\mathrm{Vect}[G]$ by a 3-coboundary $c: G^3 \longrightarrow \mathbb{C}^*$ we get a 2-algebra equivalent to $\mathrm{Vect}[G]$. Check this!

So what's really interesting is not n-cocycles but

$$H^n(G,A) = \frac{\text{n-cocycles on } G \text{ valued in } A}{\text{n-coboundaries on } G \text{ valued in } A} = \frac{Z^n(G,A)}{B^n(G,A)}$$

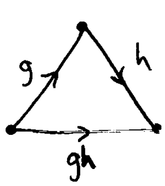
where $H^n(G,A)$ is called the nth cohomology of G with coefficients in A. This is <u>group cohomology</u>. What's good is that $H^n(G,A)$ is <u>easier</u> to calculate than $Z^n(G,A)$ & $B^n(G,A)$ — just as in topology. In fact, all your favorite topological tricks apply ... since $H^n(G,A)$ actually <u>is</u> the cohomology of some topological space ... which we now describe.

The Classifying Space BG of a Group G

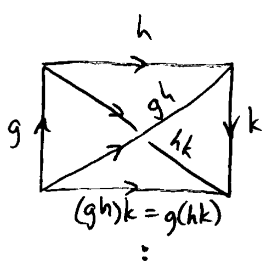
From any group G we can cook up a space with:

one vertex

one edge for each $g \in G$



one triangle for each $(g,h) \in G^2$



one tetrahedron for each (g,h,k)

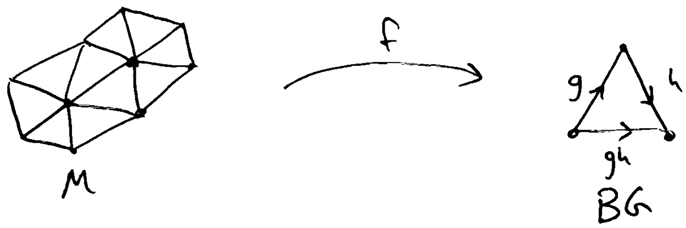
⋮

(and so on ad infinitum)

really a simplicial set.

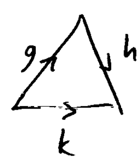
We call this space "BG". It has many remarkable properties, but I'll tell you two:

1) What's a simplicial map $f: M \longrightarrow BG$ where M is some triangulated manifold?



i.e. a map sending n-simplices to n-simplices $\forall n$.

Ans: this is just a <u>flat G-connection</u> on M!
i.e. a map sending edges of M to group elts s.t. for each triangle we have



s.t. $k = gh$

So, our TQFTs are field theories where the "fields" take values in BG! Also: here we are seeing

$$\mathcal{A}_0(M) \cong \hom(M, BG)$$

flat G-conns on M — simplicial maps

Earlier we saw

$$\mathcal{A}_0(M) \cong \hom(\Pi_1 M, G)$$

functors (between groupoids) — fundamental groupoid of M

So we've shown that

$$\hom(M, BG) \cong \hom(\Pi_1 M, G)$$

so B & Π_1 are adjoint functors. Π_1 turned topology into algebra. B turns algebra into topology.

$$\Pi_1 : [\text{simplicial sets}] \longrightarrow [\text{groupoids}]$$

So we should really have

$$B : [\text{groupoids}] \longrightarrow [\text{simplicial sets}]$$

but so far we only discussed

$$B : [\text{groups}] \longrightarrow [\text{simplicial sets}]$$

It's not hard to fix this: given a groupoid G, let BG be the space (really simplicial set!) with

$\overset{x}{\bullet}$ one vertex per object $x \in G$

$\bullet \xrightarrow{f} \bullet$ one edge per morphism $f : x \to y$ in G

& so on, with one n-simplex per composable string of n morphisms like this:

$$\underset{x_0}{\bullet} \xrightarrow{f_1} \underset{x_1}{\bullet} \xrightarrow{f_2} \underset{x_2}{\bullet} \longrightarrow \cdots \longrightarrow \bullet \xrightarrow{f_n} \bullet$$

e.g. for $n = 2$

$$\begin{tikzcd} & x_1 \arrow[dr, "f_2"] & \\ x_0 \arrow[ur, "f_1"] \arrow[rr, "f_1 f_2"'] & & x_2 \end{tikzcd}$$

Then we get adjoint functors

$$
\begin{tikzcd}
\text{[Simplicial Sets]} \arrow[r, bend left, "\Pi_1"] & \text{Groupoids} \arrow[l, bend left, "B"]
\end{tikzcd}
$$

but not an equivalence... so we're not done in our quest to unify algebra & topology.

2.) The "group cohomology" $H^n(G,A)$ is secretly the same as the "topological cohomology" $H^n(BG,A)$! This is why Eilenberg & MacLane invented BG in the first place (~1950), in order to understand group cohomology & also to compute it using ideas from topology. In our remaining short span, we'll look at BG for $G = \mathbb{Z}_2, \mathbb{Z}_p, \mathbb{Z}, \ldots$ and see that we get cool spaces whose cohomology we can (sometimes) compute!

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Computing Group Cohomology

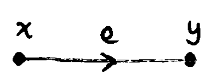
To compute $H^n(G,A)$ we'll compute $H^n(BG,A)$ (and hopefully see why they're the same). We'll do this first for $G = \mathbb{Z}_2$, starting with some beautiful pictures of the classifying space $B\mathbb{Z}_2$. For this we really need to understand BG as a simplicial set — taking degeneracies into account.

A simplicial set X consists of sets X_n of n-simplices for each $n = 0, 1, 2, \ldots$, together with various maps between these. We've seen some of these maps in our description of a "simplicial 2-graph":

$$\underset{\text{vertices}}{X_0} \;\; \begin{array}{c} \xleftarrow{\;d_0\;} \\ \xleftarrow[\;d_1\;]{} \end{array} \;\; \underset{\text{edges}}{X_1} \;\; \begin{array}{c} \xleftarrow{\;d_0\;} \\ \xleftarrow{\;d_1\;} \\ \xleftarrow[\;d_2\;]{} \end{array} \;\; X_2$$

with maps satisfying some obvious relations. A simplicial set is a generalization that allows for n-simplices of every dimension, <u>but</u> in addition to the face maps d_i there are "degeneracy maps" $e_i : X_n \to X_{n+1}$!

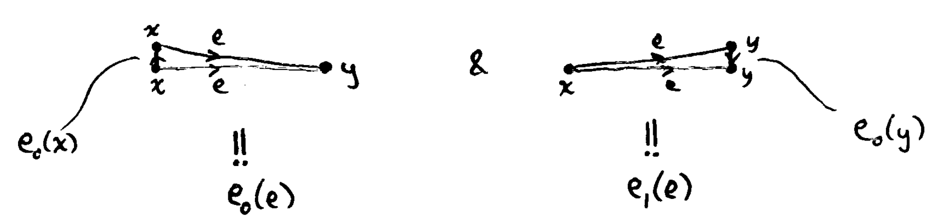
If you have, for example, an edge $e \in X_1$:



$x = d_1 e$
$y = d_0 e$

(recall: d_i <u>leaves out</u> the ith vertex)

we can get two "degenerate" triangles from it:



So what we really have in a simplicial set is

$$\begin{tikzcd} X_0 \arrow[r, bend right, "e_0"'] & X_1 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[r, bend right, "e_0"'] \arrow[r, bend right=50, "e_1"'] & X_2 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=2, "d_2"] \arrow[r, bend right, "e_0"'] \arrow[r, bend right=50, "e_1"'] \arrow[r, bend right=70, "e_2"'] & X_3 \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] & \cdots \end{tikzcd}$$

where $e_i : X_n \to X_{n+1}$ creates $(n+1)$-simplices by duplicating the ith vertex of n-simplices.

This setup explains the concept of "normalized" cocycles — those that vanish on degenerate simplices. But the <u>real</u> reason degeneracies are important is that this diagram

$$\begin{tikzcd} X_0 \arrow[r, bend right] & X_1 \arrow[l, shift right] \arrow[l, shift left] \arrow[r, bend right] \arrow[r, bend right=50] & X_2 \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] & \cdots \end{tikzcd}$$

is a picture of one of the most fundamental categories in the universe!

Namely, there's a category

Δ = [finite totally ordered sets, order preserving functions]

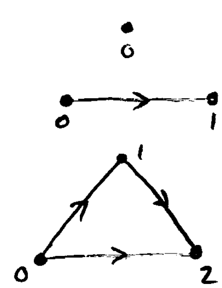
which has a skeleton with objects

$0 = \{\ \}$ — the mysterious -1-simplex

$1 = \{0\}$ — 0 simplex

$2 = \{0,1\}$ — 1 simplex

$3 = \{0,1,2\}$ — 2 simplex

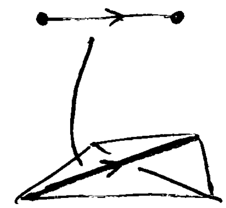


$4 = \{0,1,2,3\}$

$\vdots$

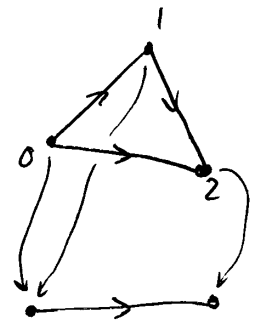
(ordered as usual!), together with order-preserving functions that can be drawn as follows:

$$\begin{tikzcd} & 0 \arrow[dl] & 1 \arrow[d] & \\ 0 & 1 & 2 & 3 \end{tikzcd}$$



But this setup automatically gives degeneracies as well, e.g.

$$\begin{tikzcd} 0 \arrow[dr] & 1 \arrow[d] & 2 \arrow[d] \\ & 0 & 1 \end{tikzcd}$$



In fact, every morphism in Δ is a composite of <u>face maps</u> $d_i : n-1 \longrightarrow n$

$$d_i : \begin{array}{ccccccccc} 0, & 1, & 2, & \dots, & i, & i+1, & \dots, & n-2 & \\ \downarrow & \downarrow & \downarrow & & \downarrow & \searrow & & \searrow & \\ 0 & 1 & 2, & \dots, & i, & i+1, & i+2, & \dots, & n-1 \end{array}$$

(which are 1-1 but not quite onto) and <u>degeneracies</u> $e_i : n+1 \longrightarrow n$

$$e_i \quad \begin{array}{ccccccc} 0, & 1, & 2, & i, & i+1, & \dots, & n \\ \downarrow & \downarrow & \downarrow & \downarrow & \swarrow & & \\ 0, & 1, & 2, & i, & \dots & & , n-1 \end{array}$$

(which are onto but not quite 1-1)

So, Δ is generated by these morphisms:

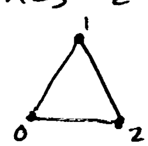
$$\begin{tikzcd} 0 \arrow[r, "d_0"] & 1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"'] & 2 \arrow[l, bend left, "e_0"] \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"'] & 3 \arrow[l, bend left, "e_0"] \arrow[l, bend left=50, "e_1"] & \cdots \end{tikzcd}$$

which looks almost like our previous picture

$$\begin{tikzcd} X_0 \arrow[r, bend right, "e_0"'] & X_1 \arrow[l, shift right=1, "d_0"'] \arrow[l, shift left=1, "d_1"] \arrow[r, bend right, "e_0"'] \arrow[r, bend right=50, "e_1"'] & X_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=3, "d_2"] & \cdots \end{tikzcd}$$

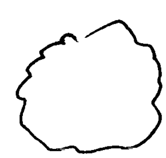
but not quite:

1) The set of n-simplices, X_n, corresponds to $n+1$, since an n-simplex has a totally ordered set of $n+1$ vertices:



2-simplex

2) We ignore the set X_{-1} corresponding to the number 0 since -1-simplices give topologists the creeps:



the -1-simplex (?)

3) All the arrows are pointing backwards.

So, we define the topologist's version of Δ to be

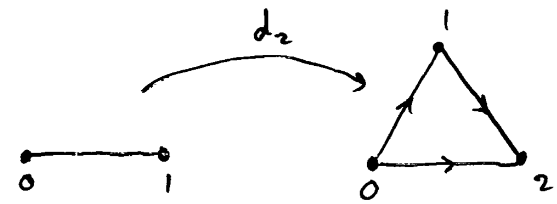
$$\Delta_0 = [\underline{\text{nonempty}} \text{ totally ordered finite sets, order preserving maps}]$$

and define a (topologist's) simplicial set to be a functor

$$X : \Delta_0^{op} \longrightarrow Set$$

assigning each $n \in \Delta_0$ a set $X(n)$ — the set of simplices with n vertices (i.e. $(n-1)$-simplices, by (1)), and to each morphism $f: n \to m$ a function $X(f) : X(m) \to X(n)$. Why the "op"? Because of remark (3).

But more fundamentally, any way to (say) include the 1-simplex in the 2-simplex:



is a morphism in Δ_0, & it gives for any simplicial set

$$X(d_2) : X(2) \longrightarrow X(1)$$

sending each triangle in $X(2)$ to its 2nd edge, an element of $X(1)$.

But for now, the main moral is that simplicial sets have degeneracy maps

$$X(e_i) : X(n) \longrightarrow X(n+1)$$

as well as face maps

$$X(d_i) : X(n) \longrightarrow X(n-1)$$

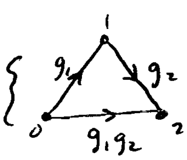
Drawing these is so tricky:

"Geometric Realization"

that it's best not to draw them at all! More importantly, when we "geometrically realize" a simplicial set and turn it into a topological space, the degenerate simplices should be realized as "squashed" i.e. lower dimensional simplices, like this:

In particular, BG is a simplicial set where

$$(BG)(n) \cong G^{n-1}$$

$(BG)_3 = \{$  $\}$

but a bunch of these $(n-1)$-simplices are <u>degenerate</u> where at least one of $(g_1, \ldots, g_{n-1}) \in G^{n-1}$ equals $1 \in G$.