8 March 2005

Computing Group Cohomology — continued.

Grothendieck used the nerve to define the classifying space of any <u>category</u>, which has the classifying space of a <u>group</u> as a special case. He called this the <u>nerve</u> of a category. Here it is: given a category C the nerve BC is the simplicial set with:

$$\{n\text{-simplices}\} = \left\{ x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_n \right\}$$

e.g.

$$\begin{tikzcd} & x_1 \arrow[dr, "f_1"] & \\ x_0 \arrow[ur, "f_0"] \arrow[rr, "f_0 f_1"'] & & x_2 \end{tikzcd}$$

and face maps:

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \cdots \xrightarrow{f_{i-1}} \bullet \xrightarrow{f_i} \bullet \cdots x_n$$

$$\Big\downarrow d_i \qquad\qquad i = 1, \dots, n-1$$

(it's a bit different for $i = 0, n$ — we leave off the first or last morphism entirely)

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \cdots x_{i-1} \xrightarrow{f_{i-1} f_i} x_{i+1} \cdots x_n$$

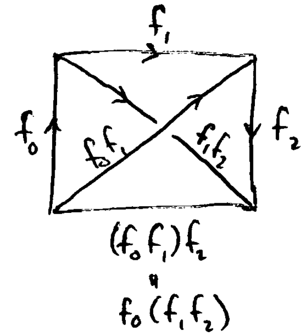
and degeneracies:

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \cdots x_{i-1} \xrightarrow{f_{i-1}} x_i \xrightarrow{f_i} \bullet \cdots x_n$$

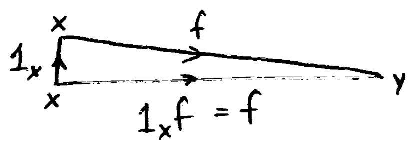
$$\Big\downarrow e_i \qquad\qquad i = 0, \dots, n$$

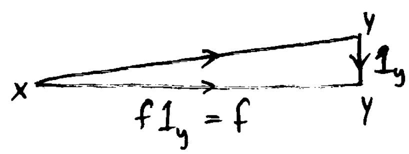
$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \cdots x_{i-1} \xrightarrow{f_{i-1}} x_i \xrightarrow{1_{x_i}} x_i \xrightarrow{f_i} x_{i+1} \cdots x_n$$

To check that BC is a simplicial set, we need to check that d_i, e_i satisfy the usual relations, using associativity:



& left/right unit laws



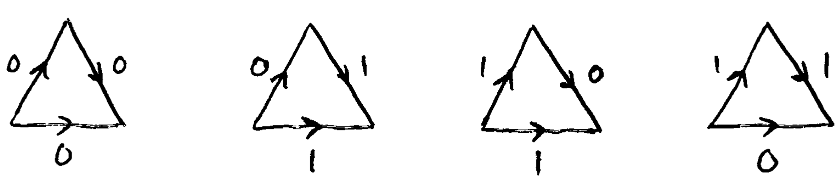


Now let's see what $B\mathbb{Z}_2$ looks like! It has one vertex:

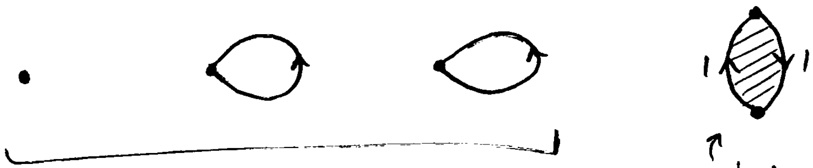
It has two edges:

but one is degenerate:

It has four triangles



but they really look like this:



these are <u>degenerate</u> triangles

↑ not degenerate (no identity on backbone)

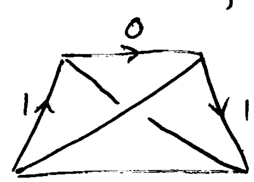
We're building up $B\mathbb{Z}_2$ one dimension at a time, & at the n-dimensional stage we get its so-called n-skeleton:

0-skeleton: $*$ $(=\mathbb{R}P^0)$

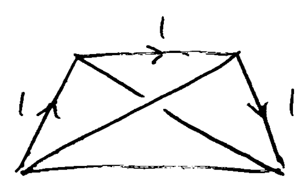
1-skeleton:  S^1 $(=\mathbb{R}P^1)$

2-skeleton:  $\mathbb{R}P^2$ (a disc with antipodal points identified — the projective plane)

Next: $B\mathbb{Z}_2$ has 8 tetrahedra, all but one having a "0" on their "backbone", hence degenerate:

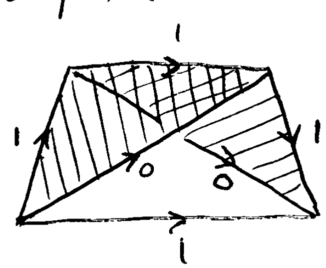


degenerate



nondegenerate

So to draw the 3-skeleton of $B\mathbb{Z}_2$ we just draw the nondegenerate 3-simplex:



and note it has only 2 nondegenerate triangular faces.

So to get the 3-skeleton of $B\mathbb{Z}_2$ we glue a 3-ball (the solid tetrahedron) along its boundary to $\mathbb{RP}^2$ and it wraps around $\mathbb{RP}^2$ twice, so we get $\mathbb{RP}^3$. I.e. we take the 3-ball & identify antipodal points.

In fact, the n-skeleton of $B\mathbb{Z}_2$ is $\mathbb{RP}^n$ &

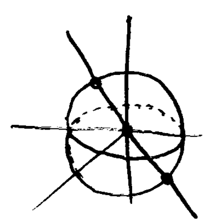
$$B\mathbb{Z}_2 = \lim_{n\to\infty} \mathbb{RP}^n$$

which is called $\mathbb{RP}^\infty$. This "limit" can be thought of as the union

$$\bigcup_{n=0}^{\infty} \mathbb{RP}^n$$

where $\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \mathbb{RP}^2 \subseteq \cdots$ in the usual way (i.e. the inclusion of the n-ball in the $(n+1)$-ball gives an inclusion $\mathbb{RP}^n \subseteq \mathbb{RP}^{n+1}$ after identifying antipodes)

We can also think of $\mathbb{RP}^n$ as the space of lines through the origin of $\mathbb{R}^{n+1}$



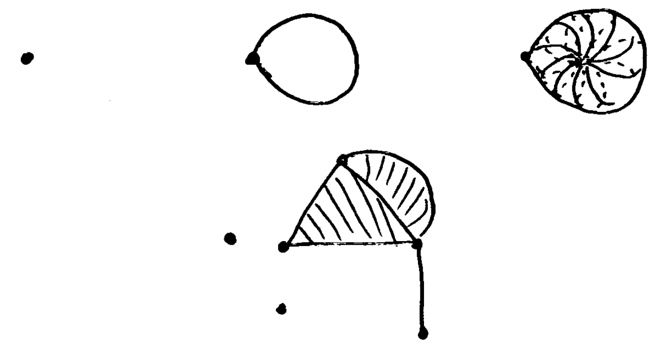
$n = 2$

which is the same as S^n with antipodal points identified, which is the same as the n-ball with antipodal points on its boundary identified: $\mathbb{RP}^n$.

So $\mathbb{RP}^\infty$ is the space of lines through the origin in $\mathbb{R}^\infty$.

Now let's calculate the group cohomology $H^n(\mathbb{Z}_2, A)$ with coefficients in some abelian group A by computing $H^n(B\mathbb{Z}_2, A)$.
It's easy to calculate the cohomology of a space X if it's a CW-complex, i.e. if it's built by taking a bunch of 0-balls, then gluing on a bunch of 1-balls along their boundaries, etc...

(note: any simplicial set is a CW complex since a simplex is a ball



If X_n is the set of these n-balls (or n-cells), a CW complex gives us numbers

$$[x, y] \in \mathbb{Z} \qquad x \in X_n,\ y \in X_{n-1}$$

saying how many times the boundary of x wraps around y (counting orientation).

To compute $H^n(X, A)$ you make up abelian groups

$$C^n(X, A) = \{ f : X_n \to A \}$$ (w. finite support!)

& maps

$$d : C^n(X,A) \longrightarrow C^{n+1}(X,A)$$

defined by

$$\underset{\substack{\Uparrow \\ X_{n+1}}}{df(x)} = \sum_{y \in X_n} [x,y] f(y)$$

(strictly speaking we only want $f: X_n \to A$ which are <u>zero</u> except on finitely many $x \in X_n$, so that this sum converges — but for $B\mathbb{Z}_2 = \mathbb{R}P^\infty$ there will be just <u>one</u> $x \in X_n$ for each n!)

In fact $d^2 = 0$ so we can define

$$H^n(X,A) = \frac{\{\ker d \subseteq C^n(X,A)\}}{\{\operatorname{im} d \subseteq C^n(X,A)\}}$$

& in fact this agrees with all other popular ways of computing the cohomology of a space X (if it's a CW complex).

For $X = \mathbb{R}P^\infty$, $A = \mathbb{Z}$:

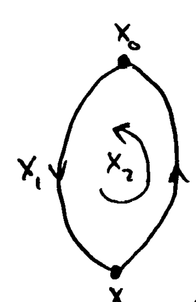
$$0 \to \underset{\substack{\cong \\ \mathbb{Z}}}{C^0(\mathbb{R}P^\infty, \mathbb{Z})} \xrightarrow[d=0]{d} \underset{\substack{\cong \\ \mathbb{Z}}}{C^1(\mathbb{R}P^\infty, \mathbb{Z})} \xrightarrow[d=2]{d} \underset{\substack{\cong \\ \mathbb{Z}}}{C^2(\mathbb{R}P^\infty, \mathbb{Z})} \xrightarrow[d=0]{d} \cdots$$

(with $\mathbb{Z}$ coeffs):

$$H^0(\mathbb{R}P^\infty) = \frac{\ker d}{\operatorname{im} d} = \frac{\mathbb{Z}}{\{0\}} = \mathbb{Z}$$

$$H^1(\mathbb{R}P^\infty) = \frac{\ker d}{\operatorname{im} d} = \frac{\{0\}}{\{0\}} = \{0\}$$

$$H^2(\mathbb{R}P^\infty) = \frac{\ker d}{\operatorname{im} d} = \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2$$



$[x_1, x_0] = 1 - 1 = 0$
$[x_2, x_1] = 1 + 1 = 2$
$[x_n, x_{n-1}]$

So:

$$H^n(\mathbb{Z}_2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n \text{ odd} \\ \mathbb{Z}_2 & n \text{ even } \& >0. \end{cases}$$

10 March 2005

Computing Group Cohomology – Concluded.

We can compute $H^n(\mathbb{Z}_2, \mathbb{C}^*) \cong H^n(\mathbb{RP}^\infty, \mathbb{C}^*)$ just as we did for $\mathbb{Z}$ coefficients, but replacing $\mathbb{Z}$'s by $\mathbb{C}^*$'s:

$$\begin{array}{ccccccccc} 1 \to C^0(\mathbb{RP}^\infty, \mathbb{C}^*) & \xrightarrow{d} & C^1(\mathbb{RP}^\infty, \mathbb{C}^*) & \xrightarrow{d} & C^2(\mathbb{RP}^\infty, \mathbb{C}^*) & \longrightarrow & C^3(\mathbb{RP}^\infty, \mathbb{C}^*) \to \cdots \\ \cong & & \cong & & \cong & & \cong \\ \mathbb{C}^* & \xrightarrow[z \mapsto 1]{} & \mathbb{C}^* & \xrightarrow[z \mapsto z^2]{} & \mathbb{C}^* & \xrightarrow[z \mapsto 1]{} & \mathbb{C}^* \end{array}$$

$H^0(\mathbb{RP}^\infty, \mathbb{C}^*) = \frac{\ker d}{\operatorname{im} d} = \frac{\mathbb{C}^*}{1} = \mathbb{C}^*$	$H^1(\mathbb{RP}^\infty, \mathbb{C}^*) = \frac{\ker d}{\operatorname{im} d} = \frac{\{\pm 1\}}{1} = \mathbb{Z}_2$	$H^2(\mathbb{RP}^\infty, \mathbb{C}^*) = \frac{\ker d}{\operatorname{im} d} = \frac{\mathbb{C}^*}{\mathbb{C}^*} = 1$	$\cdots$

So:

$$H^n(\mathbb{Z}_2, \mathbb{C}^*) = \begin{cases} \mathbb{C}^* & n=0 \\ \mathbb{Z}_2 & n \text{ odd} \\ 1 & n>0 \ \& \text{ even} \end{cases}$$

Recall: $H^n(G, \mathbb{C}^*)$ classifies ways of twisting the "Dijkgraaf-Witten" TQFT with gauge group G in n dimensions.

So we see there are *no* nontrivial ways to twist this TQFT in 2, 4, 6 ... dimensions, but *one* nontrivial way in odd dimensions.

This fact:

$$B\mathbb{Z}_2 = \mathbb{R}P^\infty$$

is part of a nice pattern:

$$\mathbb{Z}_2 = O(1) = \{1 \times 1 \text{ real orthogonal matrices}\} \implies B\mathbb{Z}_2 = \mathbb{R}P^\infty$$

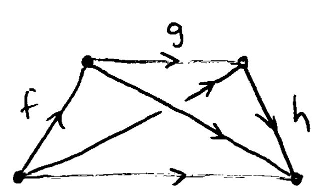
$$S^1 = U(1) = \{1 \times 1 \text{ complex unitary matrices}\} \implies BU(1) = \mathbb{C}P^\infty$$

$$SU(2) = Sp(1) = \{1 \times 1 \text{ quaternionic unitary matrices}\} \implies BSU(2) = \mathbb{H}P^\infty$$

where the latter two are classifying spaces of topological groups — where topology on G gives a topology on the set of n-simplices in BG. Someday I hope we'll study these... especially $G = SU(2)$, which show up in 3d quantum gravity.

Summary

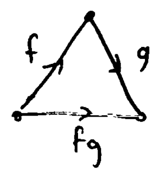
We've begun to see why it might make sense to build spacetime out of simplices... namely we describe things and processes as objects & morphisms in a category C, but C gives a simplicial set, its nerve BC:

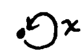


In fact, you can completely recover C from BC, using the fact that:

objects are vertices

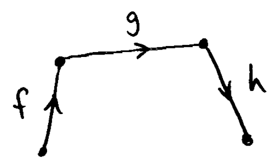
morphisms are edges

compositions are triangles 

identity morphisms are degenerate edges 

In fact a category "is" just a simplicial set with:

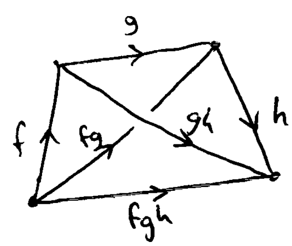
Exactly one n-simplex having any given "backbone":



backbone

$\downarrow$

! simplex



And: a functor $F: C \longrightarrow C'$ is just a simplicial map from BC to BC', so categories can be treated as simplicial sets with an extra <u>property</u>.

More generally, Ross Street has defined weak n-categories to be simplicial sets with an extra property (depending on n).

But we have concentrated on the case where C is a mere finite group G...

If M is any simplicial set, thought of as spacetime, we saw that a functor

$$A : \Pi_1 M \longrightarrow G$$

is just a flat G-connection on M, which maps (homotopy classes of) <u>paths</u> in M to group elements:

$$\longmapsto \quad g \in G$$

Alternatively, a flat connection is a simplicial map

$$\hat{A} : M \longrightarrow BG$$

since we have adjoint functors

$$\begin{tikzcd} \text{[simplicial sets]} \arrow[r, bend left, "\Pi"] & \text{[groupoids]} \arrow[l, bend left, "B"] \end{tikzcd}$$

(or even categories)

Furthermore, natural transformations:

$$\begin{tikzcd} \Pi_1 M \arrow[r, bend left=50, "A"{name=U}] \arrow[r, bend right=50, "A'"'{name=D}] & G \arrow[from=U, to=D, Rightarrow, "g"] \end{tikzcd}$$

are gauge transformations.

If M is a triangulated n-manifold, we got an n-dimensional TQFT called the "Dijkgraaf-Witten model":

$$Z : n\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

where

$$Z(M) : Z(S) \longrightarrow Z(S')$$

is computed as a sum over flat G-connections on M, which for the case of M a <u>closed</u> manifold was just:

$$Z(M) = \frac{|\mathcal{A}_0(M)|}{|\mathcal{G}(M)|} \cdot |G|^{\chi(M)} \in \mathbb{C}$$

$$= \sum_{\substack{A : M \to BG \\ \text{i.e. } [A] \in \mathcal{A}_0(M)/\mathcal{G}(M)}} \frac{1}{|\mathrm{Aut}\, A|} \cdot |G|^{\chi(M)}$$

flat conns mod gauge transformations

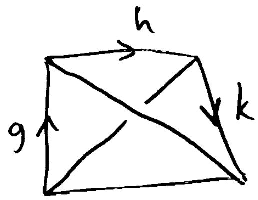
group of gauge transformations mapping A to itself

i.e. the "groupoid cardinality" of $\hom(M, BG)$ (or $\hom(\Pi_1 M, G)$) (w. flat conns. as objects, gauge transformations as morphisms) times $|G|^{\chi(M)}$.

So: The amplitude for spacetime to be M is the "number of ways" of interpreting its edges as group elements (viz a functor).

gpd. card.

Then we learned how to "twist" this TQFT by an n-cocycle $c: G^n \longrightarrow \mathbb{C}^*$, which gives the amplitude for any simplex in M to be labelled by group elements in some way:



has amplitude $c(g,h,k) \in \mathbb{C}^*$

where the cocycle condition came from demanding triangulation independence.

An n-cocycle gives an element

$$[c] \in H^n(BG, \mathbb{C}^*)$$

but a flat connection A gives a map

$$\hat{A}: M \longrightarrow BG$$

So we get

$$\hat{A}^*[c] \in H^n(M, \mathbb{C}^*)$$

by composing $\hat{A}$ & c. Here $\hat{A}^* c$ tells us an amplitude for any n-simplex in spacetime, w. edges labelled by elts of G via $\hat{A}$

(note c is like e^{iS})

In fact, we get

$$Z_{\text{twisted}}(M) = \sum_{[A] \in \mathcal{A}_0(M)/\mathcal{G}(M)} \frac{\prod_{\Delta \in M_n} c(\hat{A}\Delta)}{|\mathrm{Aut}(A)|} \, |G|^{\chi(M)}$$

where M_n is the set of n-simplices in M, & $\hat{A}$ maps each $\Delta \in M_n$ to an n-simplex in BG, which c gives an amplitude for.

Physicists should imagine

$$c(\hat{A}\Delta) = e^{iS}$$

where S is the <u>action</u> of a simplex labelled by group elts. via $\hat{A}$.

Next: what about when $G = SU(2)$???

tune in next quarter...