Exercises

1. Establish the following isomorphisms in any bicartesian closed category:

$$A + 0 \cong A, \quad A \times 0 \cong 0, \quad A^0 \cong 1;$$
$$A + B \cong B + A, \quad (A + B) + C \cong A + (B + C);$$
$$(A + B) \times C \cong (A \times C) + (B \times C), \quad A^{B+C} \cong A^B \times A^C.$$

2. Write down explicit equations between arrows to replace E5 and E6, that is, eliminate f, g and h.

3. Give a detailed justification for Definition 8.2, as was done for $p \wedge q$ in the text.

4. Show that in a bicartesian closed category $0^A \cong 0$ if and only if $A \not\cong 0$.

9 Natural numbers objects in cartesian closed categories

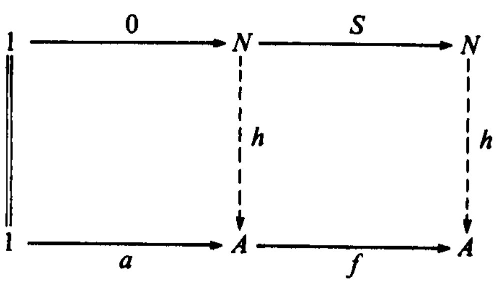
A *natural numbers object* in a cartesian closed category $\mathscr{A}$, according to Lawvere, consists of an initial object

$$1 \xrightarrow{0} N \xrightarrow{S} N$$

in the category of all diagrams $1 \xrightarrow{a} A \xrightarrow{f} A$ in $\mathscr{A}$. This means that, for all such diagrams there is a unique arrow $h: N \to A$ such that

$$h0 = a \quad \text{and} \quad hS = fh,$$

as is illustrated by the following commutative diagram:



Sometimes we merely wish to assert the existence of h, never mind its uniqueness. Then we shall speak of a *weak natural numbers object*. Cartesian closed categories with a weak natural numbers object have been called 'prerecursive categories' by Marie-France Thibault. (See Exercise 2 below.)

For example, in **Sets**, the set $\mathbb{N} \equiv \{0, 1, 2, \ldots\}$ of natural numbers together with the successor function $S(x) \equiv x + 1$ forms a natural numbers object.

More generally, all toposes considered in Part II have natural numbers objects.

If $1 \xrightarrow{0} N \xrightarrow{S} N$ is a weak natural numbers object, we shall write $h \equiv J_A(a, f)$. Thus

$$J_A: \mathrm{Hom}(1, A) \times \mathrm{Hom}(A, A) \to \mathrm{Hom}(N, A)$$

satisfies the equations

$$J_A(a, f)0 = a, \quad J_A(a, f)S = fJ_A(a, f).$$

Proposition 9.1. If the cartesian closed category $\mathscr{A}$ has a natural numbers object (weak natural numbers object) and if $x: 1 \to A$ is an indeterminate arrow over $\mathscr{A}$, then the cartesian closed category $\mathscr{A}[x]$ has the same natural numbers object (weak natural numbers object).

Proof. A short conceptual argument goes as follows. A (weak) natural numbers object in $\mathscr{A}$ gives rise to one in the slice category $\mathscr{A}/A$, hence in $\mathscr{A}[x]$, which comes with a full and faithful functor $K_x: \mathscr{A}[x] \to \mathscr{A}/A$. For the more meticulous reader, we shall now give a detailed computational proof.

First, assume the existence of a weak natural numbers object in $\mathscr{A}$. Let $\beta(x): 1 \to B$ and $\varphi(x): B \to B$ be given polynomials in $\mathscr{A}[x]$. We seek a polynomial $\chi(x): N \to B$ such that

$$\chi(x)0 =_x \beta(x), \quad \chi(x)S =_x \varphi(x)\chi(x).$$

In view of functional completeness, these equations involving x are equivalent to the following equations not involving x:

$$\kappa_{x \in A}(\chi(x)0) = \kappa_{x \in A}\beta(x), \quad \kappa_{x \in A}(\chi(x)S) = \kappa_{x \in A}(\varphi(x)\chi(x)).$$

Writing

$$\kappa_{x \in A}\beta(x) = b: A \times 1 \to B, \quad \kappa_x\varphi(x) = f: A \times B \to B,$$

we seek

$$\kappa_{x \in A}\chi(x) = h: A \times N \to B$$

such that

$$h\langle \pi, 0\pi' \rangle = b, \quad h\langle \pi, S\pi' \rangle = f\langle \pi, h \rangle.$$

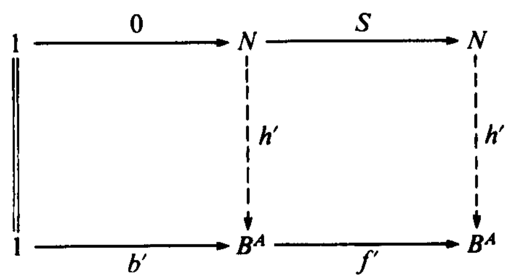
With b and f we may associate $b': 1 \to B^A$ and $f': B^A \to B^A$ given by

$$b' = (b\langle \pi', \pi \rangle)^*, \quad f' = (f\langle \pi', \varepsilon \rangle)^*.$$

Then we may find $h': N \to B^A$ such that

$$h'0 = b', \quad h'S = f'h',$$

as is illustrated by the following diagram:



Now put $h = \varepsilon\langle h'\pi', \pi\rangle$, then routine calculations show that

$$h\langle\pi, 0\pi'\rangle = b, \quad h\langle\pi, S\pi'\rangle = f\langle\pi, h\rangle,$$

as required.

If we have a natural numbers object in $\mathscr{A}$, not just a weak one, then the arrow $h': N \to B^A$ is uniquely determined by the equations $h'0 = b'$ and $h'S = f'h'$. From this it easily follows that h is also uniquely determined by the equations it satisfies. For we may calculate h' in terms of h as follows:

$$\begin{aligned}(h\langle\pi', \pi\rangle)^* &= (\varepsilon\langle h'\pi', \pi\rangle\langle\pi', \pi\rangle)^* \\ &= (\varepsilon\langle h'\pi, \pi'\rangle)^* \\ &= h',\end{aligned}$$

and then transform the equations satisfied by h into the equations satisfied by h'.

In what follows we shall write, for indeterminate arrows $y: 1 \to B$, $v: 1 \to B^B$ and $z: 1 \to N$,

$$J_B(y, v^f)z \equiv I_B\langle y, v, z\rangle,$$

where $\langle y, v, z\rangle$ is short for $\langle\langle y, v\rangle, z\rangle$.

Corollary 9.2 A weak natural numbers object in a cartesian closed category is given by an object N and arrows $1 \xrightarrow{0} N \xrightarrow{S} N$ and I_B: $(B \times B^B) \times N \to B$, for each object B, satisfying the identities

$$I_B\langle y, v, 0\rangle = y, \quad I_B\langle y, v, Sz\rangle = v^f I_B\langle y, v, z\rangle,$$

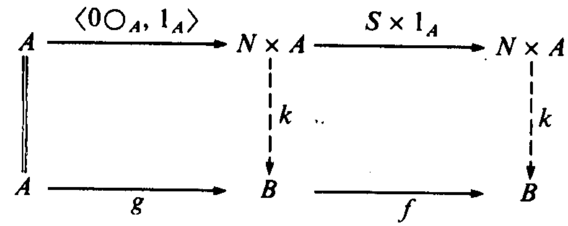
where the subscripts y, v, z on the equality symbol have been omitted.

Proof. We use Proposition 9.1 with $A = B \times B^B$. Adjoining a single indeterminate $x: 1 \to A$ is equivalent to adjoining two indeterminates $y: 1 \to B$, and $v: 1 \to B^B$.

Corollary 9.2 may also be stated in terms of an arrow $N \to (B^B)^{(B^B)}$ in place of I_B.

Exercises

1. If a cartesian closed category has a natural numbers object, then this is unique up to isomorphism.
2. Determine all weak natural numbers objects in the category of sets.
3. Carry out the routine calculations mentioned in the text to show that
   $h\langle\pi, 0\pi'\rangle = b$, $h\langle\pi, S\pi'\rangle = f\langle\pi, h\rangle$.
4. Show that a natural numbers object in a cartesian closed category is equivalent to the following condition: for each $g: A \to B$ and $f: B \to B$ there is a unique $k: N \times A \to B$ such that the following diagram commutes:



   The same assertion without uniqueness holds for a weak natural numbers object. This suggests how to define (weak) natural numbers objects in cartesian categories which are not cartesian closed.
5. Show that, if a cartesian closed category $\mathscr{A}$ has a (weak) natural numbers object, then so does $\mathscr{A}/A$ for each object A of $\mathscr{A}$.
6. Verify the remark in the proof of Proposition 9.1 that, if A is an object of a cartesian closed category $\mathscr{A}$ and $x: 1 \to A$ an indeterminate arrow, there is a full and faithful functor $K_x: \mathscr{A}[x] \to \mathscr{A}/A$.
7. (Lawvere) Given a category $\mathscr{A}$, let $\mathscr{A}^{\text{loop}}$ be the category whose objects are 'endomaps' $f: A \to A$ and whose arrows are commutative squares. There is an obvious forgetful functor $U: \mathscr{A}^{\text{loop}} \to \mathscr{A}$.
   (a) Show that $\mathscr{A}^{\text{loop}}$ is equivalent to $\mathscr{A}^{\mathbb{N}}$, where $\mathbb{N}$ is regarded as the free monoid on one generator.
   (b) Assuming that $\mathscr{A}$ is cartesian closed, show that U has a left adjoint F if and only if $\mathscr{A}$ has a natural numbers object. (Hint: In one direction define $F(A) \equiv S \times 1_A: N \times A \to N \times A$ and use Exercise 4 above. In the other direction consider $F(1)$.)

8. Consider the cartesian closed category of limit spaces (Part 0, Section 7, Example 7.8). Show that the natural numbers object is the set $\mathbb{N}$ of natural numbers with the 'discrete' convergence structure: a sequence $\{x_n | n \in \mathbb{N}\}$ converges to $x \in \mathbb{N}$ if it is eventually constant with value x.

10 Typed λ-calculi

The purpose of this section is to associate a language with a cartesian closed category with weak natural numbers object, which will be called its 'internal language'. The kind of language we have in mind will be called a 'typed λ-calculus' for short, although it might be known from the literature more fully as a 'typed $\lambda\eta$-calculus with product types (surjective pairing) and iterator'. This association will turn out to be an equivalence between appropriate categories.

A *typed λ-calculus* is a formal theory defined as follows. It consists of classes of 'types', 'terms' of each type, and 'equations' between terms which are said to 'hold', all subject to certain closure conditions. We shall write $a \in A$ to say that a is a term and is of type A; the symbol $\in$ belongs to the metalanguage.

(a) *Types*: The class of types contains two basic types and is closed under two operations as follows:

(a1) 1 and N are types (these are the 'basic' types).

(a2) If A and B are types so are $A \times B$ and B^A.
There may be other types not indicated by (a1) and (a2) and there may be un-expected identifications between types.

(b) *Terms*: The class of terms is freely generated from variables and certain basic constants by certain term forming operations as follows:*

(b1) For each type A there are countably many variables of type A, say $x_i^A \in A \quad (i = 0, 1, 2, \ldots)$. We shall hardly have occasion to refer to a specific variable, instead we shall frequently use the phrase 'let x be a variable of type A', abbreviated as '$x \in A$'.

(b2) $* \in 1$.

(b3) If $a \in A, b \in B$ and $c \in A \times B$, then $\langle a, b \rangle \in A \times B$, $\pi_{A,B}(c) \in A$ and $\pi'_{A,B}(c) \in B$.

(b4) If $f \in B^A$ and $a \in A$, then $\varepsilon_{B,A}(f, a) \in B$.

(b5) If $x \in A$ and $\varphi(x) \in B$, then $\lambda_{x\in A}\varphi(x) \in B^A$.

(b6) $0 \in N$; if $n \in N$, then $S(n) \in N$.

(b7) If $a \in A, h \in A^A$ and $n \in N$, then $I_A(a, h, n) \in A$.

* There may be other constants and term forming operations than those specified.

We shall abbreviate $\varepsilon_{B,A}(f, a)$ as $f^{\prime}a$ (read: 'f of a') when the type subscripts are clear from the context. There may be other terms not indicated by (b1) to (b7). Intuitively, $\varepsilon_{B,A}$ means evaluation, $\langle -,- \rangle$ means pairing and $\lambda_{x\in A}\varphi(x)$ denotes the function $x \mapsto \varphi(x)$. $\lambda_{x\in A}$ acts like a quantifier, so the variable x in $\lambda_{x\in A}(\varphi(x)$ is 'bound' as in $\forall_{x\in A}\varphi(x)$ or $\int_b^a f(x)dx$. We have the usual conventions for free and bound variables and when it is permitted to substitute a term for a variable. The term a is *substitutable* for x in $\varphi(x)$ if no free occurrence of a variable in a becomes bound in $\varphi(a)$. A term is 'closed' if it contains no free variables. We usually omit subscripts in $\pi_{A,B}(-)$, $I_A(-,-,-)$ etc.

(c) *Equations*:

(c1) Equations have the form $a =_X a'$, where X is a finite set of variables, a and a' have the same type A, and all variables occurring freely in a or a' are elements of X.

(c2) The binary relation between terms a, a' which says that $a =_X a'$ holds is reflexive, symmetric and transitive and it satisfies the rule: when $X \subseteq Y$ then if $a =_X b$ holds one may infer that $a =_Y b$ holds, which will be abbreviated:

$$\frac{a =_X b}{a =_Y b}.$$

It also satisfies the usual substitution rules for all term forming operations, in particular the following:

$$\frac{a =_X b}{f^{\prime}a =_X f^{\prime}b}, \quad \frac{\varphi(x) =_{X\cup\{x\}} \varphi'(x)}{\lambda_{x\in A}\varphi(x) =_X \lambda_{x\in A}\varphi'(x)},$$

from which the other substitution rules follow.

All these are 'obvious' substitution rules, except perhaps the rule involving λ, which decreases the number of free variables.

(c3) The following specific equations hold:

$a =_X *$ for all $a \in 1$;

$\pi(\langle a, b \rangle) =_X a$ for all $a \in A, b \in B$,

$\pi'(\langle a, b \rangle) =_X b$ for all $a \in A, b \in B$,

$\langle \pi(c), \pi'(c) \rangle =_X c$ for all $c \in A \times B$;

$\lambda_{x\in A}\varphi(x)^{\prime}a =_X \varphi(a)$ for all $a \in A$ which are substitutable for x,

$\lambda_{x\in A}(f^{\prime}x) =_X f$ for all $f \in B^A$, provided x is not in X (hence does not occur freely in f);

$I(a,h,0) \underset{X}{=} a,$ for all $a\in A, h\in A^A$

$I(a,h,S(x)) \underset{X\cup\{x\}}{=} h^{\mathsf{f}} I(a,h,x)$; provided x is not in X (hence does not occur freely in a or h);

$\lambda_{x\in A}\varphi(x) \underset{X}{=} \lambda_{x'\in A}\varphi(x'),$ if x' is substitutable for x.

There may be other equations not indicated by (c1) to (c3).

The last equation listed under (c3) may be omitted if we are willing to identify terms which differ only in the choice of bound variables.

One of the rules listed under (c2) allows us to pass from $a \underset{X}{=} b$ to $a \underset{X\cup\{x\}}{=} b$, even when x is not in X. Under certain conditions one can go in the opposite direction, as we shall see. Of course, if this were always the case, there would have been no point in putting the subscript X on the equality sign.

Proposition 10.1. In any typed λ-calculus, one may infer from $\varphi(x) \underset{X\cup\{x\}}{=} \psi(x)$ that $\varphi(a) \underset{X}{=} \psi(a)$ for any $a\in A$, provided x is not in X and all variables occurring freely in a are elements of X.

Proof. From $\varphi(x) \underset{X\cup\{x\}}{=} \psi(x)$ we infer $\lambda_{x\in A}\varphi(x) \underset{X}{=} \lambda_{x\in A}\psi(x)$, hence $\lambda_{x\in A}\varphi(x)^{\mathsf{f}} a \underset{X}{=} \lambda_{x\in A}\psi(x)^{\mathsf{f}} a$, using (c2). In view of (c3), we then obtain $\varphi(a) \underset{X}{=} \psi(a)$.

Corollary 10.2. If f and g do not contain free occurrences of the variable x of type A, then from $f \underset{X\cup\{x\}}{=} g$ we infer $f \underset{X}{=} g$, provided there exists a term a of type A such that all variables occurring freely in a are elements of X.

Proof. If x is not already in X, this follows from Proposition 10.1.

Unfortunately, it may happen that A is 'empty' that is, no closed term of type A exists (see examples 10.5 and 10.6, below). On the other hand, if there are closed terms of each type, the proviso of Corollary 10.2 is always satisfied. This is the case, for example, in the 'pure typed λ-calculus with weak natural numbers object' to be discussed presently. In such a situation the subscript X on $\underset{X}{=}$ is redundant and one may replace $\underset{X}{=}$ by just $=$.

Sometimes one may argue differently, but with the same result. Suppose $f \underset{X\cup\{x\}}{=} g$, then $f^{\mathsf{f}}x \underset{X\cup\{x\}}{=} g^{\mathsf{f}}x$, hence $\lambda_{x\in A}(f^{\mathsf{f}}x) \underset{X}{=} \lambda_{x\in A}(g^{\mathsf{f}}x)$. In view of (c3), it follows that $f \underset{X}{=} g$. The assumption here is that f and g have type B^A. We shall sum this up:

Proposition 10.3. If f and g are of type B^A and if $x\in A$ does not occur freely in f or g, then from $f \underset{X\cup\{x\}}{=} g$ one may infer $f \underset{X}{=} g$.

We shall consider three examples of typed λ-calculi with weak natural numbers object.

Example 10.4. Suppose there are no types, terms and equations other than those indicated by the closure rules (and also no nontrivial identifications between types), then we obtain the *pure typed λ-calculus with weak natural numbers object* called $\mathscr{L}_0$.

Example 10.5. Given a graph $\mathscr{G}$, the λ-calculus $\Lambda(\mathscr{G})$ *generated* by $\mathscr{G}$ is defined as follows. Its types are generated inductively by the type forming operations $(-)\times(-)$ and $(-)^{(-)}$ from the basic types 1, N and the vertices of $\mathscr{G}$ (which now count as basic types). Its terms are generated inductively from the basic terms x_i^A, 0 and $*$ by the old term forming operations $\langle -,-\rangle$, $\pi(-)$, $\pi'(-)$, $\varepsilon(-,-)$, $\lambda_{x\in A}(-)$, $S(-)$ and $I(-,-,-)$ together with the new term forming operations: if $a\in A$ then $fa\in B$, for each arrow $f: A\to B$ in $\mathscr{G}$. Finally, its equations are precisely those which follow from (c1) to (c3) and no others. Note that there are plenty of empty types, for instance, all the vertices of $\mathscr{G}$. Clearly, Example 10.5 includes 10.4 if $\mathscr{G}$ is the empty graph.

We now come to the principal example of this section.

Example 10.6. The *internal language* $\mathbf{L}(\mathscr{A})$ of a cartesian closed category $\mathscr{A}$ with weak natural numbers object is defined as follows. Its types are the objects of $\mathscr{A}$, with 1, N, $A\times B$ and B^A having the obvious meanings. Terms of type A are those polynomial expressions $\varphi(x_1,\ldots,x_n): 1\to A$ in the indeterminates $x_i: 1\to A_i$ which are obtained from variables, namely indeterminates, and basic constants, namely arrows $1\to A$ in $\mathscr{A}$, by the term forming operations:

$$\frac{a: 1\to A \quad b: 1\to B}{\langle a,b\rangle: 1\to A\times B},\ \frac{a: 1\to A}{fa: 1\to B},\ \frac{\varphi(x): 1\to B}{\lambda_{x\in A}\varphi(x): 1\to B^A},$$

where $f: A\to B$ and $\lambda_{x\in A}\varphi(x) \equiv \ulcorner\kappa_{x\in A}\varphi(x)\langle 1_A, \bigcirc_A\rangle\urcorner$ as in the proof of Corollary 6.2. Moreover, we write $*$ for $\bigcirc_1$, $\pi_{A,B}(c)$ for $\pi_{A,B}c$, $\varepsilon_{B,A}(f,a)$ for $\varepsilon_{B,A}\langle f,a\rangle$, etc. Finally, if a and b are polynomial expressions whose free variables are in X, $a \underset{X}{=} b$ is said to hold in $\mathbf{L}(\mathscr{A})$ if $a \underset{X}{=} b$ as polynomials in $\mathscr{A}[X]$.

We shall now introduce morphisms $\Phi: \mathscr{L}\to\mathscr{L}'$ of typed λ-calculi, to be called *translations*.

(d1) Φ sends types of $\mathscr{L}$ to types of $\mathscr{L}'$ and terms of $\mathscr{L}$ to terms of $\mathscr{L}'$ so that if $a\in A$, then $\Phi(a)\in\Phi(A)$; but we insist that if a is closed, so is $\Phi(a)$ and that Φ sends the ith variable of type A to the ith variable of type $\Phi(A)$.

(d2) Φ preserves the specified type operations on the nose, for example:

$$\Phi(1)=1,\quad \Phi(A\times B)=\Phi(A)\times\Phi(B),\ldots;$$

and the specified term forming operations up to 'equality holding', e.g. the following equations hold in $\mathscr{L}'$:

$$\Phi(\pi_{A,B}(c))=\pi_{\Phi(A),\Phi(B)}(\Phi(c));\quad \Phi(\lambda_{x\in A}\varphi(x))=\lambda_{\Phi(x)\in\Phi(A)}\Phi(\varphi(x)).$$

(d3) Moreover, Φ preserves equations:

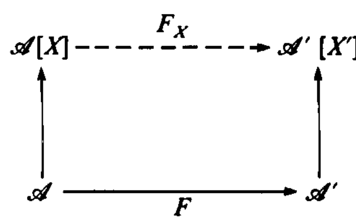
if $a \underset{X}{=}$ holds in $\mathscr{L}$ then $\Phi(a) \underset{\Phi(X)}{=} \Phi(b)$ holds in $\mathscr{L}'$.

In view of (d3), Φ really acts on equivalence classes of terms (modulo the equivalence relation described in (c2)). We shall say that two translations are *equal* if they have the same effect on such equivalence classes. Thus $\Phi=\Psi$ provided $\Phi(a) \underset{\Phi(X)}{=} \Psi(a')$ holds whenever $a \underset{X}{=} a'$ holds.

We thus obtain a category **λ-Calc** whose objects are typed λ-calculi and whose arrows are translations.

Let $\mathbf{Cart}_{\mathrm{N}}$ be the category of cartesian closed categories with weak natural numbers object and cartesian closed functors preserving weak natural numbers objects on the nose. The proof of the following is left to the reader.

Proposition 10.7. Let $\mathbf{L}(\mathscr{A})$ be the internal language of $\mathscr{A}$ and, for any morphism $F:\mathscr{A}\to\mathscr{A}'$, let $\mathbf{L}(F)$ be defined by $\mathbf{L}(F)(A)=F(A)$, $\mathbf{L}(F)(x_i)=x_i'$, $\mathbf{L}(F)(\varphi(X))=F_X(\varphi(X))$, where x_i' is the ith variable of type $F(A)$ and F_X is the unique arrow in $\mathbf{Cart}_{\mathrm{N}}$ such that the following diagram commutes:



Then $\mathbf{L}$ is a functor from $\mathbf{Cart}_{\mathrm{N}}$ to **λ-Calc.**

$\mathscr{L}_0$ is an initial object in **λ-Calc**, that is, for any typed λ-calculus $\mathscr{L}$ there is a unique translation $\mathscr{L}_0\to\mathscr{L}$. In particular, for any $\mathscr{A}$ in $\mathbf{Cart}_{\mathrm{N}}$ there is a unique morphism $\mathscr{L}_0\to\mathbf{L}(\mathscr{A})$. This may be called the *interpretation* of $\mathscr{L}_0$ in $\mathscr{A}$.

The reader may have noticed that the languages discussed in this section may be proper classes in the sense of Gödel–Bernays. If necessary, one may work in a set theory with universes, in which 'classes' are replaced by 'sets in a sufficiently large universe'.

Exercises

1. Verify that Λ: **Grph** → **λ-Calc** (see Example 10.5) is a functor left adjoint to the obvious 'forgetful' functor V: **λ-Calc** → **Grph.** (The underlying graph of a λ-calculus has as vertices the types and as arrows $A\to B$ suitable equivalence classes of pairs $(x,\varphi(x))$, where $\varphi(x)$ is a term of type B with no free variables other than x, which is of type A.)
2. By a *classification* we mean two classes and a mapping between them:

The mapping assigns to each entity its type, and we write '$a\in A$' for 'the type of a is A'. Morphisms Φ between classifications are defined in the obvious way: $a\in A$ should imply $\Phi(a)\in\Phi(A)$. The category of small classifications is thus equivalent to $\mathbf{Sets}^{\mathbf{2}}$, where **2** is the category consisting of two objects and one arrow between them. Show that the obvious forgetful functor from **λ-Calc** to the category of classifications has a left adjoint.

3. If $\mathscr{C}$ is a Heyting algebra considered as a cartesian closed category, show that there may be unexpected identifications between types in $\mathbf{L}(\mathscr{C})$.
4. Verify that $\mathbf{L}(\mathscr{A})$ in Example 10.6 is a typed λ-calculus.

11 The cartesian closed category generated by a typed λ-calculus

To show that the functor $\mathbf{L}$ in Section 10 is an equivalence of categories we shall obtain a functor $\mathbf{C}$ in the opposite direction.

Given a typed λ-calculus $\mathscr{L}$, we construct a cartesian closed category $\mathbf{C}(\mathscr{L})$ with weak natural numbers object as follows:

The objects of $\mathbf{C}(\mathscr{L})$ are the types of $\mathscr{L}$.

The arrows $A\to B$ of $\mathbf{C}(\mathscr{L})$ are (equivalence classes of) pairs $(x\in A,\varphi(x))$, with x a variable of type A and $\varphi(x)$ a term of type B with no free variables other than x. (Think of the function $x\mapsto\varphi(x)$.)

Equality of arrows is defined by: $(x\in A, \varphi(x)) = (x'\in A, \psi(x'))$ if and only if $\varphi(x) \underset{x}{=} \psi(x)$ holds, where $\underset{x}{=}$ abbreviates $\underset{\{x\}}{=}$.

The identity arrow $A \to A$ is the pair $(x\in A, x)$.

The composition of $(x\in A, \varphi(x))\colon A\to B$ and $(y\in B, \psi(y))\colon B \to C$ is given by $(x\in A,\ \psi(\varphi(x)))\colon A \to C$, $\varphi(x)$ having been substituted for y in $\psi(y)$.

The cartesian closed structure of $\mathbf{C}(\mathscr{L})$ is obtained as follows:

$$\bigcirc_A \equiv (x\in A, *),$$
$$\pi_{A,B} \equiv (z\in A \times B, \pi(z)),$$
$$\pi'_{A,B} \equiv (z\in A \times B, \pi'(z)),$$
$$\langle (z\in C, \varphi(z)), (z\in C, \psi(z))\rangle \equiv (z\in C, \langle \varphi(z), \psi(z)\rangle),$$
$$(z\in A \times B, \chi(z))^* \equiv (x\in A, \lambda_{y\in B}\chi(\langle x, y\rangle)),$$
$$\varepsilon_{C,A} \equiv (y\in C^A \times A, \varepsilon_{C,A}(\pi(y), \pi'(y))).$$

$\mathbf{C}(\mathscr{L})$ has a weak natural numbers object:

$$0 \equiv (x\in 1, 0),$$
$$S \equiv (x\in N, S(x)),$$
$$I_B \equiv (w\in (B \times B^B) \times N, I(\pi(\pi(w)), \pi'(\pi(w)), \pi'(w))).$$

It is easy to make $\mathbf{C}$ into a functor λ-**Calc** $\to$ $\mathbf{Cart}_\mathbf{N}$. Indeed, suppose $\Phi\colon \mathscr{L} \to \mathscr{L}'$ is a translation, define $\mathbf{C}(\Phi)\colon \mathbf{C}(\mathscr{L}) \to \mathbf{C}(\mathscr{L}')$ as follows:

If A is an object of $\mathbf{C}(\mathscr{L})$, that is, a type of $\mathscr{L}$, $\mathbf{C}(\Phi)\ (A) = \Phi(A)$ is the corresponding type of $\mathscr{L}'$, hence an object of $\mathbf{C}(\mathscr{L}')$.

If $f = (x\in A,\ \varphi(x))$ is an arrow $A \to B$ in $\mathbf{C}(\mathscr{L})$, that is, $\varphi(x)$ is a term of type B in $\mathscr{L}$, $\mathbf{C}(\Phi)\ (f) = \Phi(x)\in\Phi(A)$, $\Phi(\varphi(x))$ is the corresponding arrow $\Phi(A) \to \Phi(B)$ in $\mathbf{C}(\mathscr{L}')$.

To sum up:

Proposition 11.1. $\mathbf{C}$ is a functor from λ-**Calc** to $\mathbf{Cart}_\mathbf{N}$.

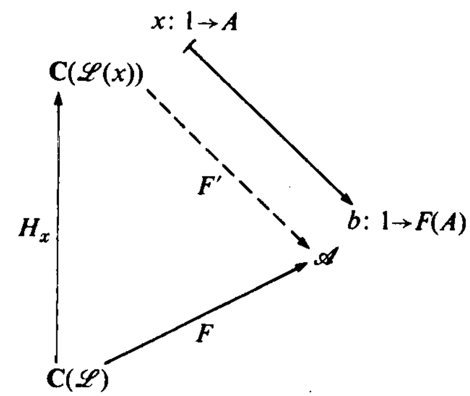
Instead of adjoining an indeterminate arrow $x\colon 1 \to A$ to the cartesian closed category $\mathbf{C}(\mathscr{L})$, one may adjoin a 'parameter' x of type A to the language $\mathscr{L}$. To be precise, if $\mathscr{L}$ is a typed λ-calculus and x is a variable of type A, one may form another language $\mathscr{L}(x)$ by *adjoining the parameter* x as follows:

$\mathscr{L}(x)$ has exactly the same types as $\mathscr{L}$ and also the same terms, except that x is no longer counted as a variable. In other words, the closed terms of $\mathscr{L}(x)$ are terms $\varphi(x)$ in $\mathscr{L}$ which contain no free variables other than x. In the same spirit, $\underset{X}{=}$ in $\mathscr{L}(x)$ means $\underset{X\cup\{x\}}{=}$ in $\mathscr{L}$: just make sure that x is not in X.

Some dictionaries define a 'parameter' as a 'variable constant'. For us it is a variable kept constant.

Proposition 11.2. $\mathbf{C}(\mathscr{L})[x] \cong \mathbf{C}(\mathscr{L}(x))$.

Proof. We show that $\mathbf{C}(\mathscr{L}(x))$ has the universal property of $\mathbf{C}(\mathscr{L})[x]$ (see Section 5):



The indeterminate $x\colon 1 \to A$ is defined by $(y\in 1,\ x)$. H_x is $\mathbf{C}$ of the inclusion of $\mathscr{L}$ into $\mathscr{L}(x)$, which may necessitate some relabelling of variables. Suppose $F\colon \mathbf{C}(\mathscr{L}) \to \mathscr{A}$ is any cartesian closed functor preserving the weak natural numbers object, and given any arrow $b\colon 1 \to F(A)$ in $\mathscr{A}$, we claim that there is a unique such functor $F'\colon \mathbf{C}(\mathscr{L}(x)) \to \mathscr{A}$ such that $F'H_x = F$ and $F'(x) = b$.

Indeed, put $F'(B) = F(B)$ for each object B of $\mathbf{C}(\mathscr{L})$, that is type in $\mathscr{L}$. Suppose $f = (y\in B, \varphi(x, y))$ is any arrow $B \to C$ in $\mathbf{C}(\mathscr{L}(x))$, that is, $\varphi(x, y)$ is any term of type C in $\mathscr{L}$ with free variables $x\in A$ and $y\in B$. Define $F'(f)$: $F(B) \to F(C)$ in $\mathscr{A}$ as follows:

First note that $\varphi(x, y) \underset{\{x,y\}}{=} \psi\lambda(y)^{\,\mathsf{f}}\, x$ holds, where $\psi(y)$ is $\lambda_{x\in A}\varphi(x, y)$. Thus $f = \varepsilon_{C,A}\langle g, x\bigcirc_B\rangle$, where $g = (y\in B, \psi(y))\colon B \to C^A$ in $\mathbf{C}(\mathscr{L})$. Now define

$$F'(f) = \varepsilon_{F(C),F(A)}\langle F(g), b\bigcirc_{F(B)}\rangle.$$

That this definition has the right property is easily seen. Moreover, it is clearly forced upon us, since $F'(g) = F(g)$ and $F'(x) = b$. We now establish the main result of this section.

Theorem 11.3. The categories λ-**Calc** and $\mathbf{Cart}_\mathbf{N}$ are equivalent, in fact $\mathbf{CL} \cong \mathrm{id}$ and $\mathbf{LC} \cong \mathrm{id}$.

Proof. (i) Consider the natural transformation $\varepsilon: \mathbf{CL} \to \mathrm{id}$ defined for each $\mathscr{A}$ in $\mathbf{Cart_N}$ by $\varepsilon(\mathscr{A}): \mathbf{CL}(\mathscr{A}) \to \mathscr{A}$ as follows:

An object of $\mathbf{CL}(\mathscr{A})$ is a type of $\mathbf{L}(\mathscr{A})$, that is, an object of $\mathscr{A}$. Put $\varepsilon(\mathscr{A})(A) = A$.

An arrow $B \to C$ in $\mathbf{CL}(\mathscr{A})$ has the form $f = (y \in B, \varphi(y))$, where $\varphi(y) \in C$ in $\mathbf{L}(\mathscr{A})$. Put $\varepsilon(\mathscr{A})\ (f) =$ the unique arrow $g: B \to C$ such that $gy =_y \varphi(y)$, using functional completeness.

It is easily verified that $\varepsilon(\mathscr{A})$ is an arrow in $\mathbf{Cart_N}$. Moreover, in view of functional completeness, it establishes a one-to-one correspondence between $\mathrm{Hom}_{\mathbf{CL}(\mathscr{A})}(B, C)$ and $\mathrm{Hom}_{\mathscr{A}}(B, C)$. Thus $\varepsilon(\mathscr{A})$ is an isomorphism.

(ii) Consider the natural transformation $\eta: \mathrm{id} \to \mathbf{LC}$ defined for each $\mathscr{L}$ in λ-**Calc** by $\eta(\mathscr{L}): \mathscr{L} \to \mathbf{LC}(\mathscr{L})$ as follows:

$$\eta(\mathscr{L})(A) \equiv A;$$

$$\eta(\mathscr{L})(\varphi(x_1, \ldots, x_n)) \equiv (z \in 1, \varphi(x_1, \ldots, x_n)) \text{ in } \mathbf{C}(\mathscr{L}(x_1, \ldots, x_n)).$$

Note that we have identified $\mathbf{C}(\mathscr{L})[x_1, \ldots, x_n]$ with $\mathbf{C}(\mathscr{L}(x_1, \ldots, x_n))$ as is justified by Proposition 11.2. It is easily verified that $\eta(\mathscr{L})$ is an arrow in λ-**Calc.** To see that $\eta(\mathscr{L})$ is an isomorphism, construct its inverse, which sends $(z \in 1, \varphi(z))$ onto $\varphi(*)$.

Corollary 11.4. $\mathbf{C}(\mathscr{L}_0)$, the free cartesian closed category with weak natural numbers object generated by the pure typed λ-calculus, is an initial object in $\mathbf{Cart_N}$.

The initial object of $\mathbf{Cart_N}$ may also be obtained by the methods of Section 4.

We end this section with a remark concerning the problem of how to interpret languages in categories. In the present context this is explained quite easily: an *interpretation* of a typed λ-calculus $\mathscr{L}$ in a cartesian closed category $\mathscr{A}$ with weak natural numbers object is just a translation $\mathscr{L} \to \mathbf{L}(\mathscr{A})$. By Theorem 11.3 (or just by adjointness, see Exercise 3 below), this is essentially the same as a cartesian closed functor $\mathbf{C}(\mathscr{L}) \to \mathscr{A}$. As already observed after Proposition 10.7, $\mathscr{L}_0$ has a unique interpretation in any cartesian closed category with weak natural numbers object.

Exercises

1. Show how to obtain the free cartesian closed category with a weak natural numbers object generated by any classification. (See Exercise 2 of Section 10.)
2. In the spirit of this section, find a new method for constructing the free cartesian closed category with a weak natural numbers object generated by a graph.
3. Show that **C** is left adjoint to **L** with adjunction η and ε.
4. Prove that $I_B\langle\langle y, v\rangle, x\rangle = (t \in 1, \quad I(y, v, x))$ in $\mathbf{C}(\mathscr{L}(y, v, x))$.

12 The decision problem for equality

Let us look at the cartesian closed category with weak natural numbers object freely generated by the empty graph, as in Section 4, but with weak natural numbers object, or as in Exercise 2 of Section 11. Since both are initial objects in $\mathbf{Cart_N}$ (see Corollary 11.4), they are isomorphic. We shall write $\mathscr{C}_0$ for this initial object. $\mathscr{C}_0$ is of interest to logicians, as it gives a version of Gödel's primitive recursive functionals of finite type, and to categorists, as it is related to the so-called 'coherence problem' for $\mathbf{Cart_N}$. This problem asks when diagrams in a category commute or, equivalently, when two arrows between two given objects are equal. Indeed if one wants to compute $\mathrm{Hom}(A, B)$ in $\mathscr{C}_0$, two problems arise:

(I) Find an algorithm for obtaining all arrows $A \to B$ in $\mathscr{C}_0$ (that is, all proofs $A \to B$ in the corresponding deductive system).

(II) Find an algorithm for deciding when two arrows $A \to B$ are equal (better: when two proofs describe the same arrow).

We shall here address ourselves to the second problem. Looking at the proof of the distributive law

$$\langle f, g\rangle h = \langle fh, gh\rangle$$

for cartesian closed categories given in Section 3, we note that both sides must be expanded to be shown equal. It seems easier to consider $\mathscr{C}_0$ as given by $\mathbf{C}(\mathscr{L}_0)$ rather than as constructed by the method of Section 4.

Two arrows $f, g: A \to B$ in $\mathscr{C}_0$ are thus given by two terms $\varphi(x)$ and $\psi(x)$ of type B in $\mathscr{L}_0$ with a free variable x of type A. We want to decide whether $\varphi(x) =_x \psi(x)$ holds, or equivalently, $\lambda_{x \in A}\phi(x) = \lambda_{x \in A}\psi(x)$ holds in $\mathscr{L}_0$. Let us call two terms a and b of $\mathscr{L}_0$ whose free variables are contained in X *convertible* if the equation $a =_X b$ holds in $\mathscr{L}_0$. Terms of $\mathscr{L}_0$ are, of course, defined inductively, as the reader will recall. Thus Problem (II) has been reduced to deciding when two closed terms of type B in $\mathscr{L}_0(x)$ or $\mathscr{L}_0$ are convertible. In view of the fact that there are closed terms of each type in $\mathscr{L}_0$, we need not distinguish between $=_x$ and $=$, as was pointed out in Section 10.

Actually, we shall solve the decision problem for convertibility not in $\mathscr{L}_0$ but in $\mathscr{L}'_0$, which is like $\mathscr{L}_0$ but without type 1. In other words, $\mathscr{L}'_0$ is a