Quantization and Cohomology

Winter 2007

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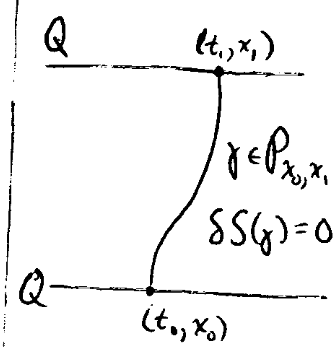
The big picture in quantum mechanics:

Classical — Lagrangian



Q = config. space

$\mathcal{P}_{x_0; x_1} = \{ \gamma : [t_0, t_1] \to Q : \gamma(t_i) = x_i \}$

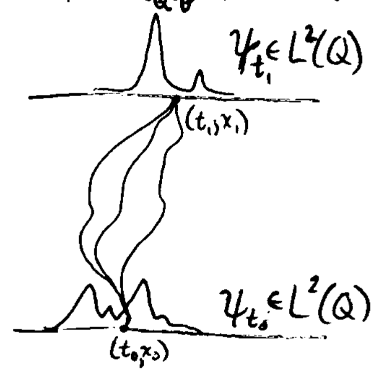
path space

$L : TQ \longrightarrow \mathbb{R}$ Lagrangian

$S = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t))\, dt$ action

Quantum — Lagrangian

$\psi_{t_1}(x_1) = \int_Q \int_{\mathcal{P}} \psi_{t_0} e^{\frac{i}{\hbar} S(\gamma)} \mathcal{D}\gamma\, dx_0$



Classical — Hamiltonian

$X = T^*Q$ = phase space

$H : X \longrightarrow \mathbb{R}$ Hamiltonian

$H = p_i \dot q^i - L \circ \lambda$

$\lambda : TQ \to T^*Q$

Legendre transform

$\tilde\gamma(t) = (q(t), p(t)) \in T^*Q$

satisfies Hamilton's eqns:

$$\frac{d}{dt} \tilde\gamma(t) = v_H(\tilde\gamma(t))$$

$$dH = \varpi(v_H, -)$$

v_H = "Hamiltonian vector field"

ϖ = "symplectic structure" on T^*Q

Quantum — Hamiltonian

$\psi_t \in L^2(Q)$ wavefunction

satisfies Schrödinger's eqn:

$$\frac{d}{dt} \psi_t = i\hbar \hat H \psi_t$$

where $\hat H : L^2(Q) \to L^2(Q)$

is a linear operator obtained by "quantizing" H.

This chart raises lots of questions:

1) How do you do the "path integral" over $\mathcal{P}_{x_0,x_1}$?
Apparently, there's no meaning to the "measure" $\mathcal{D}\gamma$, but there is to $e^{iS(\gamma)/\hbar}\mathcal{D}\gamma$, at least in well-behaved cases, e.g. the case where (Wiener measure? in S.H.O. case)

$$Q = \text{smooth fin-dim manifold}$$

$$L(q,\dot{q}) = \frac{m}{2}|\dot{q}|^2 - V(q)$$

where $|\dot{q}|$ is defined using a complete Riemannian metric. The completeness assumption is needed to keep our particle from "falling off the edge", & $V: Q \to \mathbb{R}$ should be smooth and bounded below, for the same reason.

For the basic ideas, try Feynman & Hibbs' "Quantum Mechanics and Path Integrals." For mathematical rigor, try Barry Simons' "Functional Integration and Quantum Physics."

2) How do we get the Hamiltonian operator $\hat{H}: L^2(Q) \to L^2(Q)$ from the Hamiltonian function $H: T^*Q \to \mathbb{R}$? In some cases, it's easy to write down $\hat{H}$, e.g. under the same assumptions we wrote down in question #1:

$$H(q,p) = \frac{|p|^2}{2m} + V(q)$$

w. Q a complete Riemannian manifold & $V: Q \to \mathbb{R}$ smooth and bded below.

In this situation Schrödinger wrote:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \mathrm{mult}_V$$

where ∇^2 is the Laplacian on Q and mult_V is the operator "multiplication by V," (often written simply as "V"). Schrödinger got this by guessing the quantization rule

$$p \longmapsto \frac{\hbar}{i}\vec{\nabla}$$

Under our assumptions on Q & V, Kato & Rellich showed that $\hat{H}$ is self-adjoint, which is precisely what you need to solve Schrödinger's equation. If $A: K \to K$ is a self adjoint operator on a Hilbert space, $e^{iAt}: K \to K$ is well-defined and unitary, & defining

$$\psi_t = e^{iAt}\psi_0 \qquad (\hbar = 1)$$

we get

$$\frac{d}{dt}\psi_t = iA\psi_t .$$

But we'd like a much more systematic theory of "quantizing" functions $H: T^*Q \to \mathbb{R}$ to get operators $\hat{H}: L^2(Q) \longrightarrow L^2(Q)$. Even better, can we handle the case when the phase space X isn't T^*Q? Then we don't even have $L^2(Q)$ at hand.

This leads us to "geometric quantization". For more on this, try:

http://math.ucr.edu/home/baez/quantization.html

Then try Sniatycki's book. A lot of cohomology comes into the game — starting with the fact that $[\omega] \in H^2(X,\mathbb{R})$ must ~~define~~ come from an <u>integral</u> cohomology class

$$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{R})$$