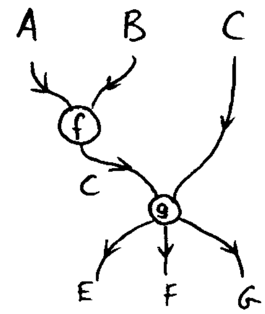
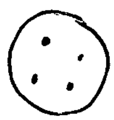
18 Jan 2007

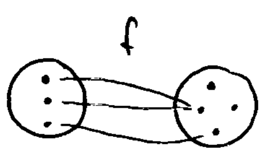
Last quarter we discussed closed monoidal categories as a framework for processes including computations, Feynman diagrams, etc ...

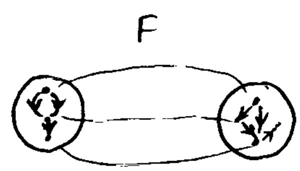


This quarter, we'll discuss closed monoidal 2-categories as a framework for discussing "processes between processes". In physics, this idea leads to string theory; it also shows up in computation.

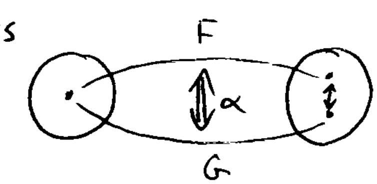
To get at 2-categories, we used a certain method called "categorification":

 sets $\longmapsto$ categories 



functions $\longmapsto$ functors


equations between functions $\longmapsto$ natural isos between functors



but we also need to dream up new equations, "coherence laws", for our natural isomorphisms. Let's do an example and categorify the concept of category...

Def: A $\begin{cases} \text{category} \\ \text{2-category} \end{cases}$ C is a collection of <u>objects</u>, together with:

1) for each pair of objects $x, y \in C$ a $\begin{cases} \text{set} \\ \text{category} \end{cases}$ $\mathrm{Hom}(x,y)$ of morphisms from x to y:

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, "g"] \arrow[r, bend right=50, "h"'] & y \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, "g" description, ""{name=G}] \arrow[r, bend right=60, "h"', ""{name=H, above}] \arrow[from=F, to=G, phantom, "\Downarrow\alpha"] \arrow[from=G, to=H, phantom, "\Downarrow\beta"] & \bullet \end{tikzcd}$$

the set $\mathrm{Hom}(x,y)$ in a category. <u>elts</u> of $\mathrm{Hom}(x,y)$ are <u>morphisms</u> in C

the category $\mathrm{Hom}(x,y)$ in a 2-category. <u>objects</u> of $\mathrm{Hom}(x,y)$ are morphisms in C; morphisms in $\mathrm{Hom}(x,y)$ are 2-morphisms in C

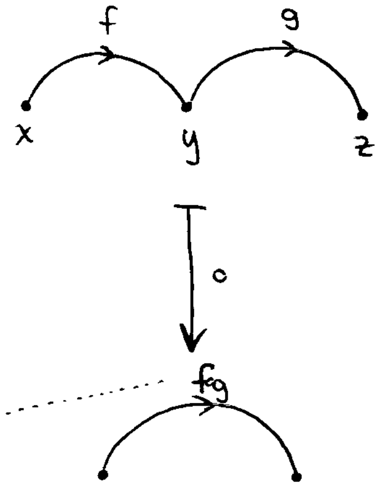
e.g. $\alpha : f \Rightarrow g$

2) For each triple of objects $x, y, z \in C$, a $\begin{cases} \text{function} \\ \text{functor} \end{cases}$ called composition:

$$\circ : \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \longrightarrow \mathrm{Hom}(x,z)$$

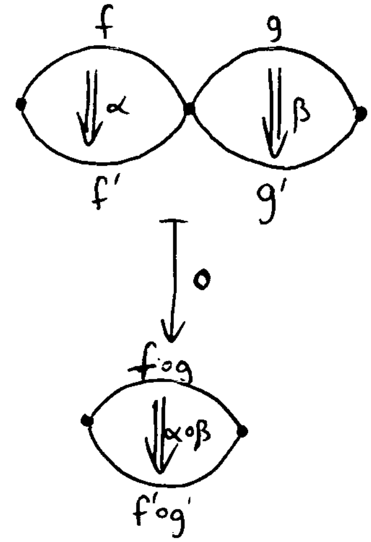
(cartesian product of $\begin{cases} \text{sets} \\ \text{categories} \end{cases}$).

In a category:

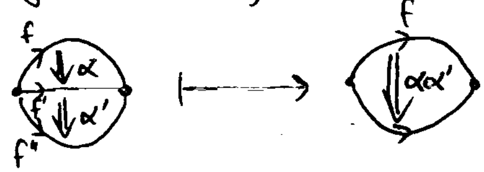


warning: last quarter we called this "gf"

In a 2-category, we have the same thing, but also:

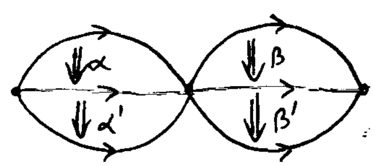


called the horizontal composition of α & β. (Since $\mathrm{Hom}(x,y)$ is a category, it already has a composition



called the vertical composition of 2-morphisms in C).

Since the functor $\circ : \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \to \mathrm{Hom}(x,z)$ preserves (vertical) composition we get an equation: given

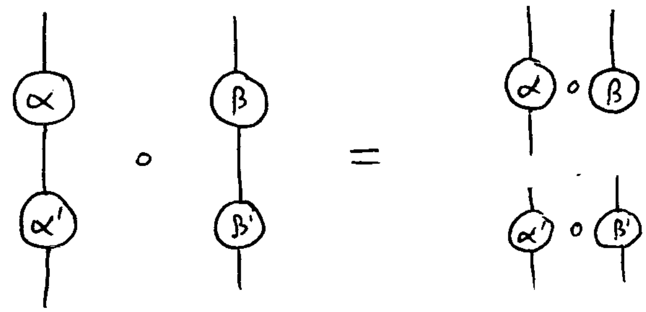


we get $(\alpha\alpha')\circ(\beta\beta') = (\alpha\circ\beta)(\alpha'\circ\beta')$ called the "middle 4 interchange law" or "exchange identity".

Aside:

Exchange law examples:

1) For a circuit:



2) In linear algebra, if we have operators

$$\alpha : V \longrightarrow V' \qquad \alpha' : V' \longrightarrow V''$$
$$\beta : W \longrightarrow W' \qquad \beta' : W' \longrightarrow W''$$

we have

$$(\alpha\alpha') \otimes (\beta\beta') = (\alpha \otimes \beta)(\alpha' \otimes \beta')$$

Back to the definition...

3) For each object $x \in C$ we have an identity (1-) morphism

$$1_x \in \mathrm{Hom}(x, x).$$

4) For each 4-tuple $w, x, y, z \in C$ we have an associativity $\left\{\begin{array}{l}\text{equation}\\ \text{natural isomorphism}\end{array}\right.$: Given $f: w \to x$, $g: x \to y$, $h: y \to z$ we have:

In a category:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

In a 2-category

$$\alpha_{f,g,h} : (f \circ g) \circ h \Rightarrow f \circ (g \circ h)$$

the associator.

5) For each $x, y \in C$ we have left & right unit $\left\{\begin{array}{l}\text{equations}\\ \text{natural isomorphisms}\end{array}\right.$. Given $f: x \to y$, these say

$$\left\{\begin{array}{l} 1_x \circ f = f = f \circ 1_y \\ 1_x \circ f \underset{\ell_f}{\overset{\sim}{\Longrightarrow}} f \underset{r_f}{\Longleftarrow} f \circ 1_y \end{array}\right.$$

left and right unitors

Now we're done defining a category, but in a 2-category we have coherence laws...

6) The pentagon identity says this diagram commutes:

$$\begin{tikzcd}[column sep=small]
 & (e\circ f)\circ(g\circ h) \arrow[dr, Rightarrow, "\alpha_{e,f,g\circ h}"] & \\
((e\circ f)\circ g)\circ h \arrow[ur, Rightarrow, "\alpha_{e\circ f,g,h}"] \arrow[d, Rightarrow, "\alpha_{e,f,g}\circ 1_h"'] & & e\circ(f\circ(g\circ h)) \\
(e\circ(f\circ g))\circ h \arrow[r, Rightarrow, "\alpha_{e,f\circ g,h}"'] & e\circ((f\circ g)\circ h) \arrow[ur, Rightarrow, "1_e\circ\alpha_{f,g,h}"'] &
\end{tikzcd}$$

7) The triangle identity says this commutes:

$$\begin{tikzcd}
(f\circ 1)\circ g \arrow[rr, Rightarrow, "\alpha_{f,1,g}"] \arrow[dr, Rightarrow, "r_f\circ 1_g"'] & & f\circ(1\circ g) \arrow[dl, Rightarrow, "1_f\circ l_g"] \\
 & f\circ g &
\end{tikzcd}$$

MacLane's Thm says these coherence laws are enough

What we're calling a 2-category is also called a bicategory, or a weak 2-category to emphasize that associativity & r/l unit laws hold up to isomorphism. A strict 2-category is one where α, l & r are all identity natural isos. These used to be called 2-categories.

Examples:

1) our main hoped-for example has:

data types as objects
programs as morphisms
processes of computation as 2-morphisms