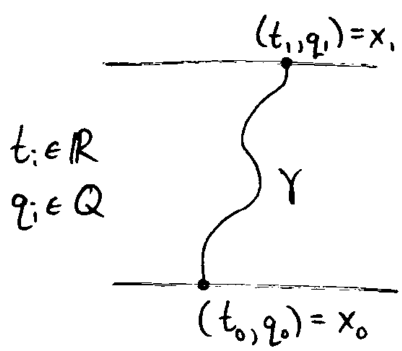
23 Jan 2007

Besides the obvious questions raised by the chart presented last time, there's a bigger question:

WHAT'S REALLY GOING ON?

Is "quantization" some arbitrary trick, or does it have some deeper meaning? Let's try to dig deeper! What sort of entity is the action? Recall in one approach we have a configuration space Q and then



$$\gamma \in \mathcal{P}_{x_0,x_1} = \{\gamma : [t_0,t_1] \longrightarrow Q : \gamma(t_i) = q_i\}$$

the action is a function $S : \mathcal{P}_{x_0,x_1} \longrightarrow \mathbb{R}$ for each $x_0, x_1 \in \mathbb{R} \times Q$. Note action respects composition of paths. Given a path $\gamma_1 \in \mathcal{P}_{x_0,x_1}$ and a path $\gamma_2 \in \mathcal{P}_{x_1,x_2}$ we can compose them to get a path $\gamma_1\gamma_2 \in \mathcal{P}_{x_0,x_2}$ and

$$S(\gamma_1\gamma_2) = S(\gamma_1) + S(\gamma_2)$$

This suggests that

THE ACTION IS A FUNCTOR

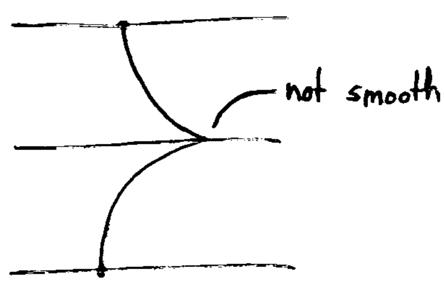
from some category $\mathcal{P}$ with

- $x = (t, q) \in \mathbb{R} \times Q$ as objects
- given objects x_0, x_1, paths $\gamma \in \mathcal{P}_{x_0, x_1}$ as morphisms

to some category $\mathbb{R}$ with

- one object
- real numbers as morphisms

Composition in $\mathcal{P}$ is composition of paths (note this is associative since we don't need to reparameterize our paths), while composition in $\mathbb{R}$ is just addition. Technical issue: we can't use smooth paths in our definition of $\mathcal{P}$ since composing paths may give nondifferentiable cusps



We could use piecewise smooth paths, though.

So: can we do classical & quantum mechanics starting with any functor

$$S : C \longrightarrow \mathbb{R}$$

where now C is any category with

- "configurations" as objects
- "paths" as morphisms. ?

How did we use $S: \mathcal{P} \to \mathbb{R}$ in the chart last time?
Classically (replacing $\mathcal{P}$ by an arbitrary C):

1) We "criticize" $S: C \to \mathbb{R}$ — i.e. for each $x, y \in C$ we look at $S: \mathrm{Hom}(x,y) \to \mathbb{R}$ and seek critical points, i.e. $\gamma \in \mathrm{Hom}(x,y)$ w. $dS(\gamma) = 0$. This only makes sense if each set $\mathrm{Hom}(x,y)$ is a manifold or a more general "infinite dimensional manifold" (e.g. a space of piecewise smooth paths in a manifold Q), and S is differentiable. This is addressed by the theory of smooth categories and smooth functors, cf:

   http://math.ucr.edu/home/baez/2conn.pdf

1') We <u>minimize</u> it — i.e. for each $x, y \in C$ we seek $\gamma \in \mathrm{Hom}(x,y)$ that minimize $S(\gamma)$.

For both (1) & (1') the issue of existence & uniqueness of γ criticizing/minimizing S is very important

Case (1′) is closer to quantum mechanics, which we can see if we study Hamilton's principal function: given $x, y \in C$ this is given by

$$Z(x,y) = \inf_{\gamma : x \to y} S(\gamma)$$

(assuming the infimum exists). In classical mechanics we get the Hamilton-Jacobi equations by differentiating $Z(x,y)$ w.r.t. x or y. These are the classical analog of Schrödinger's equation.

Now consider the quantum case:

2) We integrate its exponential — i.e. for each x, y we compute an "amplitude"

$$Z_\hbar(x,y) = \int_{\gamma \in \mathrm{Hom}(x,y)} e^{iS(\gamma)/\hbar} \, \mathcal{D}\gamma$$

For this, we want each set $\mathrm{Hom}(x,y)$ to be a measure space (or "generalized measure space") & $e^{iS(\gamma)/\hbar}$ to be integrable.

In this case we can get Schrödinger's equation by differentiating $Z_\hbar(x,y)$ w.r.t. x or y.

Now compare

$$1'): \quad Z(x,y) = \inf_{\gamma : x \to y} S(\gamma)$$

$$2): \quad Z_\hbar(x,y) = \int_{\gamma : x \to y} e^{iS(\gamma)/\hbar} \, \mathcal{D}\gamma$$

CLASSICAL	QUANTUM
$S(\gamma) \in \mathbb{R}$	$e^{iS(\gamma)/\hbar} \in \mathbb{C}$
$\inf$	$\int$
$\min$	$+$
$S(\gamma\gamma') = S(\gamma) + S(\gamma')$	$e^{iS(\gamma\gamma')/\hbar} = e^{iS(\gamma)/\hbar} e^{iS(\gamma')/\hbar}$
$+$	$\cdot$