25 Jan 2007

More Examples of 2-Categories

2) Cat – the 2-category with

- categories as objects
- functors as morphisms
- natural transformations as 2-morphisms

In particular, we can compose natural transformations both horizontally and vertically.

<u>Vertical composition</u>:

We need $\alpha\beta : F \Rightarrow H$, and in particular given $c \in C$

$$(\alpha\beta)_c : Fc \longrightarrow Hc$$

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=60, "H"'{name=H}] & D \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] \end{tikzcd}$$

We take the composite $Fc \xrightarrow{\alpha_c} Gc \xrightarrow{\beta_c} Hc$ to be $(\alpha\beta)_c$. Check that $\alpha\beta$ is natural.

Horizontal composition:

we need $\alpha \circ \beta : F \circ G \Rightarrow F' \circ G'$,
and in particular given $c \in C$

$$\begin{array}{ccc} (\alpha\circ\beta)_c : (F\circ G)c & \longrightarrow & (F'\circ G')c \\ \parallel & & \parallel \\ G(Fc) & & G'(F'c) \end{array}$$

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd}$$

α gives us

$$\alpha_c : F_c \longrightarrow F'_c$$

and the functors G & G' map this to

$$G(F_c) \xrightarrow{G(\alpha_c)} G(F'_c)$$

$$G'(F_c) \xrightarrow{G'(\alpha_c)} G'(F'_c)$$

and β gives

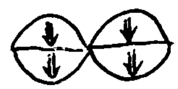
$$\begin{tikzcd} G(F_c) \arrow[r, "G(\alpha_c)"] \arrow[d, "\beta_{F_c}"'] & G(F'_c) \arrow[d, "\beta_{F'_c}"] \\ C'(F_c) \arrow[r, "G'(\alpha_c)"'] & G'(F'_c) \end{tikzcd}$$

which commutes by naturality of β. So we can use either composite to define

$$(\alpha\circ\beta)_c : (F\circ G)(c) \longrightarrow (F'\circ G')(c)$$

Check that $\alpha\circ\beta$ is natural.

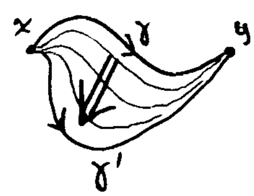
<u>Homework</u>: Check that Cat is a 2-category — a <u>strict</u> 2-category. So check:

$\alpha\beta$ is natural
$\alpha \circ \beta$ is natural
associativity & r/l unit laws for vertical and horizontal composition
interchange law 

3) For any topological space X, there's a 2-category $\Pi_2(X)$ with

- points of X as objects 

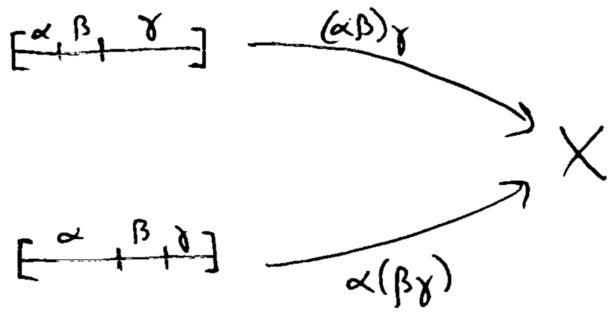
- paths in X as morphisms 

- homotopy classes of path homotopies as 2-morphisms 

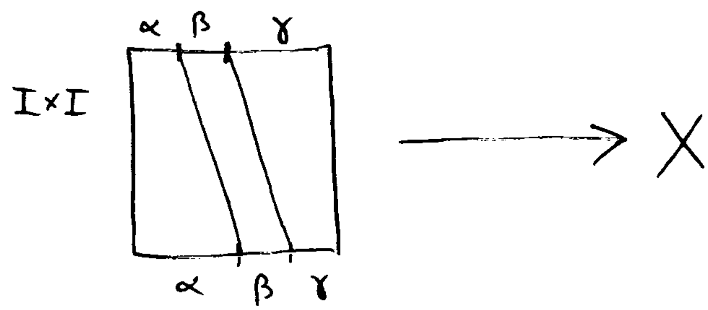
Unlike Cat, $\Pi_2(X)$ is a <u>weak</u> 2-category: given paths

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet$$

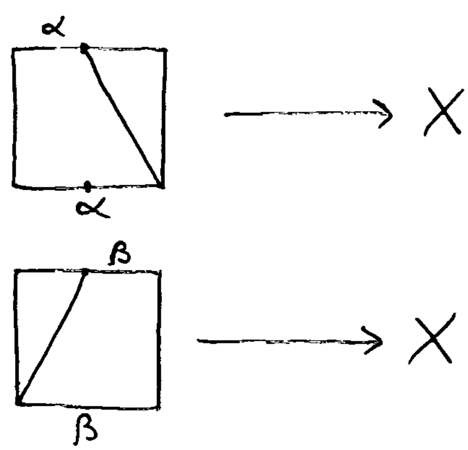
we <u>don't</u> have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$:



but we do have an associator : a 2-isomorphism from one to the other :



Similarly for l/r unitors :



Need to check the pentagon and triangle identities.

In $\Pi_2(X)$, every 2-morphism $h : \alpha \Rightarrow \beta$ has an <u>inverse</u> $h^{-1} : \beta \Rightarrow \alpha$: $hh^{-1} = 1$ & $h^{-1}h = 1$, so they're all 2-isomorphisms. Also, every morphism $\gamma : x \to y$ has a <u>weak inverse</u>, i.e. $\bar{\gamma} : y \to x$ s.t. there exist 2-isomorphisms $h : \gamma\bar{\gamma} \overset{\sim}{\Rightarrow} 1$ & $h' : \bar{\gamma}\gamma \overset{\sim}{\Rightarrow} 1$, so every 1-morphism is an <u>equivalence</u>.

(In Cat, a 2-iso. is called a <u>natural isomorphism</u> & an equivalence is called an <u>equivalence</u>)

A 2-category where every morphism is an equivalence & every 2-morphism is a 2-isomorphism is a <u>2-groupoid</u>, & $\Pi_2(X)$ is the <u>fundamental 2-groupoid</u> of X.

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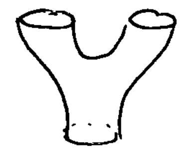
4) There's a 2-category Top_2 with

- topological spaces as objects
- continuous maps as morphisms
- homotopy classes of homotopies between maps as 2-morphisms

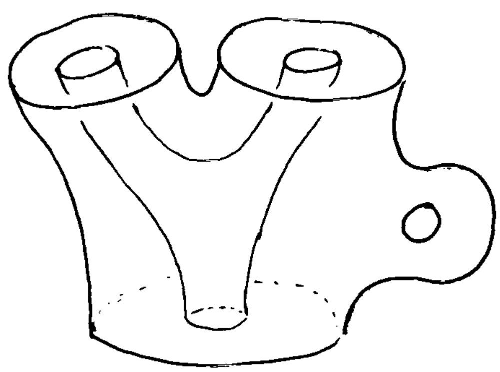
This is <u>strict</u>.

5) There's a weak 2-category $n\mathrm{Cob}_2$ for any $n \geq 2$ with (roughly)

- $(n-2)$-dimensional manifolds as objects  x
- $(n-1)$-dimensional manifolds with boundary (cobordisms) as morphisms 

$$\begin{tikzcd} x \arrow[d, "f"] \\ x' \end{tikzcd}$$

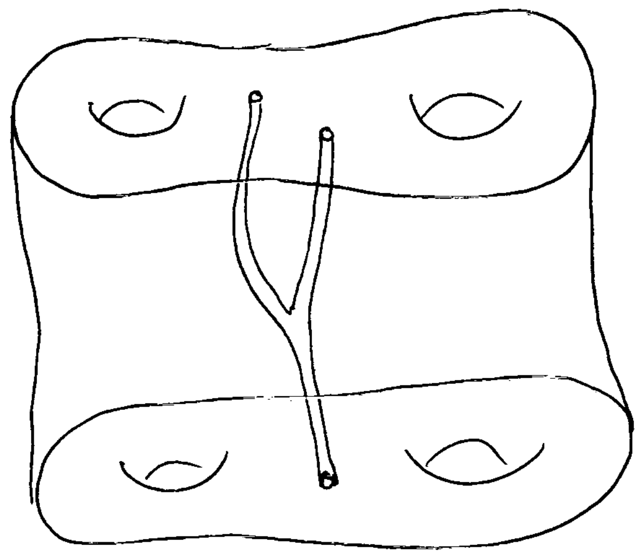
- n-dimensional manifolds with corners ("cobordisms between cobordisms") as 2-morphisms:



$$\begin{tikzcd} X \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ X' \end{tikzcd}$$

Jeffrey Morton has constructed this in his paper "A Double Bicategory of Cobordisms with Corners." In physics, the 2-morphisms here represent choices of <u>spacetime</u>, 1-morphisms represent choices of <u>space</u>, 0-morphisms represents manifolds that could be the <u>boundary</u> of space.

If $n=3$, those boundaries (unions of circles) act like particles, which can interact:



In this framework, a (once) <u>extended topological quantum field theory</u> is a nice 2-functor

$$Z : n\mathrm{Cob}_2 \longrightarrow 2\mathrm{Hilb}$$

just as an ordinary TQFT is a nice functor

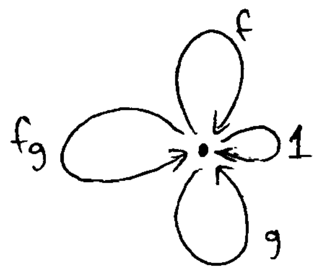
$$Z : n\mathrm{Cob} \longrightarrow \mathrm{Hilb}$$

where $n\mathrm{Cob}$ is a mere 1-category with

- $(n-1)$-dim manifolds as objects 
- n-dim cobordisms as morphisms  $(n=2)$

6) In string theory we need a 2-category $2Cob_2^{\mathbb{C}}$ which is like $2Cob_2$ but where 2-morphisms (2-manifolds w. corners) have a complex analytic structure. This is challenging to define.

7) A monoidal category is the same as a 2-category with one object, just as a monoid is the same as a category with one object:



Given a 2-category with just one object, we do a relabelling game:

2 morphisms $\rightsquigarrow$ morphisms

morphisms $\rightsquigarrow$ objects

one object $\rightsquigarrow$ —— (forget about it!)

$\underbrace{\hspace{3cm}}_{\text{2-category w. 1 object}}$ $\underbrace{\hspace{3cm}}_{\text{category where we can}}$

2-category
w. 1 object

category where we can "tensor" (compose) objects and "tensor" (horizontally compose) morphisms

E.g. the monoidal category (Vect, $\otimes$) can be considered as a (weak) 2-category & interchange law says

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "V"] \arrow[r, "V'" description] \arrow[r, bend right=60, "V''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & \bullet \arrow[r, bend left=60, "W"] \arrow[r, "W'" description] \arrow[r, bend right=60, "W''"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \bullet
\end{tikzcd}$$

$$(\alpha \otimes \beta)(\alpha' \otimes \beta') = (\alpha\alpha') \otimes (\beta\beta')$$

For any field k, this (Vect, $\otimes$) with base field k sits inside a (weak) 2-category Bim w:

- rings as objects
- (R, R')-bimodules as morphisms from R to R'.
- (R, R')-bimodule homomorphisms as 2-morphisms

(So, in the monoidal category (Vect, $\otimes$), considered as a 2-category with one object, it is nice to think of the object as the base field)