30 Jan 2007

Classical, Quantum, & Statistical Mechanics as "Matrix Mechanics"

We've seen a big analogy:

CLASSICAL MECHANICS OF POINT PARTICLES	QUANTUM MECHANICS OF POINT PARTICLES
action $S \in \mathbb{R}^{\min} = \mathbb{R} \cup \{+\infty\}$	amplitude $e^{iS/\hbar} \in \mathbb{C}$
$\min$	$+$
$+\infty$ $\quad (x \min +\infty \equiv x)$	0 $\quad (x + 0 \equiv x)$
$(\mathbb{R}^{\min}, \min, +\infty)$ is a commutative monoid	$(\mathbb{C}, +, 0)$ is an abelian group
$+$	$\cdot$
0	1
$(\mathbb{R}^{\min}, \min, +\infty, +, 0)$ is a rig	$(\mathbb{C}, +, 0, \cdot, 1)$ is a ring

we minimize action and integrate amplitudes

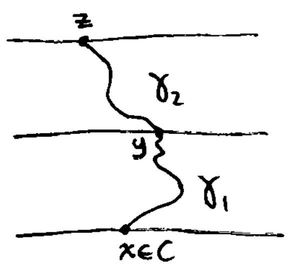
when we compose paths we add their action & multiply amplitudes

A rig is a "ring without negatives", i.e. a commutative monoid under addition, a monoid under multiplication, satisfying l/r distributivity. $\mathbb{R}^{\min}$ is a rig that's not a ring, but satisfies the idempotent law $x \min x \equiv x$.

In classical mechanics, action is a functor

$$S : C \longrightarrow \mathbb{R}^{min}$$

where C is any category (whose objects are called "configurations" & morphisms are called "paths") & $\mathbb{R}^{min}$ is a category with one object whose morphisms are $x \in \mathbb{R}^{min}$ & composition is $+$:

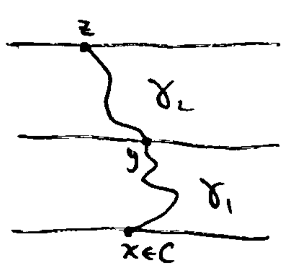


$$S(\gamma_1 \gamma_2) = S(\gamma_1) + S(\gamma_2)$$

In quantum mechanics, amplitude is a functor

$$e^{iS/\hbar} : C \longrightarrow \mathbb{C}$$

where $\mathbb{C}$ is a category with one object whose morphisms are $x \in \mathbb{C}$ & composition is $\cdot$:



$$e^{iS/\hbar}(\gamma_1 \gamma_2) = e^{iS/\hbar}(\gamma_1) \cdot e^{iS/\hbar}(\gamma_2)$$

Example: ~~Let~~ $C = \mathcal{P}$, the category whose:

- objects are points $x = (t,q) \in \mathbb{R} \times Q$ where Q is some "configuration space" (manifold).
- morphisms $\gamma : (t_0, q_0) \longrightarrow (t_1, q_1)$ are paths $\gamma : [t_0, t_1] \longrightarrow Q$ with $\gamma(t_i) = q_i$

In the quantum case, a wavefunction

$$\psi : Q \longrightarrow \mathbb{C}$$

tells us the amplitude for a particle to be at $q \in Q$. We describe the time evolution of ψ by:

$$\psi(t_1, q_1) = \int_{q_0 \in Q} \int_{\gamma : [t_0, q_0] \to [t_1, q_1]} e^{iS(\gamma)/\hbar} \, \psi(t_0, q_0) \, \mathcal{D}\gamma \, dq_0$$

So, classically, a wavefunction

$$-i\hbar \ln \psi : Q \longrightarrow \mathbb{R}^{\min}$$

$$\xrightarrow{\text{quantization}}$$
$$S \longmapsto e^{iS/\hbar}$$
$$-i\hbar \ln \psi \longmapsto \psi$$

tells us the action for a particle to be at $q \in Q$. This $-i\hbar \ln \psi$ is a known entity in classical mechanics; let's call it ψ_c for short. By our analogy, it should evolve in time as follows:

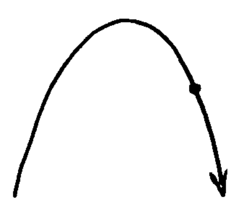
$$\psi_c(t_1, q_1) = \inf_{q_0 \in Q} \inf_{\gamma : [t_0, q_0] \to [t_1, q_1]} S(\gamma) + \psi_c(t_0, q_0)$$

If we imagine $\psi_c(t_0, q_0)$ as the cost to start a trip at $q_0 \in Q$ at time t_0, $S(\gamma)$ as the cost of the trip γ, this formula tells us $\psi_c(t_1, q_1)$ is the cheapest price to be at $q_1 \in Q$ at time t_1. In the quantum case you can go ahead and compute $\frac{d\psi}{dt}$ using the path integral — the result is <u>Schrödinger's equation</u>. In the classical case you get the <u>Hamilton–Jacobi equation</u>.

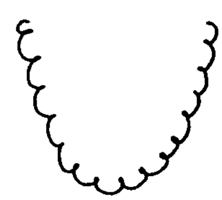
Last quarter we saw the dynamics of particles is analogous to the statics of strings. This analogy involves "Wick rotation", the substitution

$$t \longmapsto -it$$

For example:



rock in gravitational field



spring in gravitational field

$$S = \int_{t_0}^{t_1} \frac{m}{2}\frac{dq}{dt}\cdot\frac{dq}{dt} - V(q(t))\,dt \quad \xmapsto[t \mapsto -it]{} \quad S = \int_{-it_0}^{-it_1} -\frac{m}{2}\frac{dq}{dt}\cdot\frac{dq}{dt} - V(q(-it))\,d(-it)$$

but in the second case we write

$$E = -iS = \int_{t_0}^{t_1} \frac{m}{2}\dot{q}\cdot\dot{q} + V(q) \; dt$$

since this is the energy of the spring, where m is now the tension of the spring & energy is "tension energy + gravitational energy."