Statistical Mechanics & Deformation of Rigs

Last time we saw that the classical mechanics (dynamics) of particles becomes the classical statics of strings by doing the substitutions

$$t \longmapsto -it$$
$$S \longmapsto +iE$$

Minimizing the action becomes minimizing the energy. What does the quantum mechanics (dynamics) of particles become when we do these substitutions?

In quantum mechanics, the relative amplitude for a particle to trace out a path is

$$e^{iS/\hbar}$$

In "statistical mechanics" (really thermal statics — classical statics but with nonzero temperature T), the relative probability for a system to be in a configuration of energy E is:

$$e^{-E/kT}$$

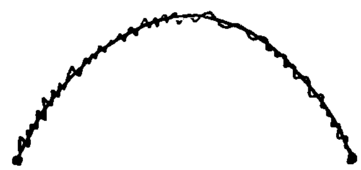
where k is Boltzmann's constant (a conversion factor between energy and temperature). Note: we have

$$e^{iS/\hbar} \longmapsto e^{-E/kT}$$

if we make the substitutions

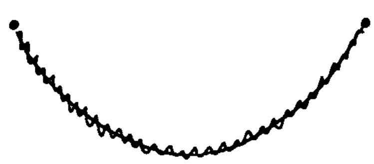
$$S \longmapsto iE$$
$$\hbar \longmapsto kT$$

This makes sense, since $\hbar$ measures how big "quantum fluctuations" are:



"quantum version of a thrown rock"

while kT measures how big "thermal fluctuations" are:



"hanging spring at nonzero temperature"

From now on, let's set $k=1$ & use the substitution

$$\hbar \longmapsto T.$$

Note that

$$e^{iS/\hbar} \in \mathbb{C}, \text{ the rig of relative amplitudes}$$
$$e^{-E/T} \in \mathbb{R}^+, \text{ the rig of relative probabilities}$$

Here $\mathbb{R}^+$ is the rig

$$([0,\infty), +, 0, \cdot, 1)$$

($e^{-E/T} \in [0,\infty)$ allows for the case $E=\infty$. We assume $T>0$)

In short we have:

$$\text{PARTICLES} \xrightarrow[S \mapsto iE]{t \mapsto it} \text{STRINGS}$$

Classical dynamics	Classical statics
action $S \in \mathbb{R}^{\min}$	energy $E \in \mathbb{R}^{\min}$
Quantum dynamics	Thermal statics
relative amplitude $e^{iS/\hbar} \in \mathbb{C}$	relative probability $e^{-E/T} \in \mathbb{R}^+$

$$\text{Quantum dynamics} \xrightarrow{\hbar \mapsto T} \text{Thermal statics}$$

We'd like to understand how quantum mechanics reduces to classical mechanics as $\hbar \to 0$ but it's easier to understand how thermal statics reduces to classical statics as $T \to 0$.

To do this, we'll formulate thermal statics using E instead of $e^{-E/T}$: for any $T>0$, we consider the Boltzmann map:

$$\begin{aligned} \beta_T : \mathbb{R}^{min} &\longrightarrow \mathbb{R}^+ \\ E &\longmapsto e^{-E/T} \\ \infty &\longmapsto 0 \end{aligned}$$

This isn't a rig homomorphism, just a 1-1 onto function. So, we'll pull back the rig structure on $\mathbb{R}^+$ to $\mathbb{R}^{min}$ via β_T and get a rig $\mathbb{R}^T$. As a set $\mathbb{R}^T$ is just $[0,\infty)$, but now it's a rig with:

$$\begin{aligned} a +_T b &= \beta_T^{-1}(\beta_T(a) + \beta_T(b)) \\ 0_T &= \beta_T^{-1}(0) \\ a \cdot_T b &= \beta_T^{-1}(\beta_T(a) \cdot \beta_T(b)) \\ 1_T &= \beta_T^{-1}(1) \end{aligned}$$

__HW:__ Work out $+_T, 0_T, \cdot_T, 1_T$ explicitly and show

$$\begin{aligned} \lim_{T\to 0} a +_T b &= a \min b \\ \lim_{T\to 0} 0_T &= +\infty \\ \lim_{T\to 0} a \cdot_T b &= a+b \\ \lim_{T\to 0} 1_T &= 0 \end{aligned}$$

(So: "the to topological rig $\mathbb{R}^T$ converges to the topological rig $\mathbb{R}^{min}$ as $T\to 0$")

The moral is: thermal statics reduces to classical statics as $T \to 0$; in both we're really doing linear algebra over some rig, & $\mathbb{R}^T \to \mathbb{R}^{min}$ as $T \to 0$.

Alas, seeing classical mechanics as an $\hbar \to 0$ limit of quantum mechanics is harder, since

$$\begin{aligned} \beta_\hbar : \mathbb{R}^{min} &\longrightarrow \mathbb{C} \\ S &\longmapsto e^{iS/\hbar} \\ +\infty &\longmapsto 0 \end{aligned}$$

is neither 1-1 nor onto & its image isn't a subrig (though it's closed under multiplication) so we can't pull the rig structure on $\mathbb{C}$ back to $\mathbb{R}^{min}$. But people do study quantization indirectly using the $\lim_{T \to 0} \mathbb{R}^T = \mathbb{R}^{min}$ idea, which is called:

- tropical mathematics (a really stupid term for some particular work by Brazilian mathematicians — "arctic mathematics" would be better for $T \to 0$ math.)
- idempotent analysis (since $a \min a = a$)
- Maslov dequantization (in reference to how $T \to 0$ lets us study the $\hbar \to 0$ limit)