Categorifying the λ-calculus

Last time we saw that a presentation of some set-based algebraic structure, e.g. a monoid, also gives rise to a category if we interpret the relations not as equations but rewrite rules. We get a category with:

- objects being expressions built from the generators using the operations in our structure.
- morphisms being built from the rewrite rules using composition and the operations in our structure.

Now we'll do something similar where our algebraic structure will be "a typed λ-calculus" as defined by Lambek & Scott (without their "natural numbers object"). Lambek & Scott show that a typed λ-calculus can serve as a presentation of a cartesian closed category (CCC). A CCC is already a category-based algebraic structure, so if we treat Lambek & Scott's relations as rewrite rules, we'll get a 2-categorical structure: <u>hopefully</u> a "cartesian closed 2-category". In such a thing, we should have:

- objects being types $A, B, C, \ldots$
  including product types $A \times B, \ldots$
  function types $\hom(A,B), \ldots$

- morphisms $f: A \longrightarrow B$ are pairs $(x \in A, \varphi(x))$ where $\varphi(x)$ is a λ-term of type B including the variable x as a free variable.

e.g. $$\varphi(x) = \underbrace{y \in \hom(A,C) \longmapsto y(x)}_{\substack{\text{a } \lambda\text{-term of type} \\ \hom(\hom(A,C),C)}}$$

So for each $x \in A$, the pair $(x, \varphi(x))$ is a morphism

$$f : A \longrightarrow \hom(\hom(A,C),C)$$

- 2-morphisms are built from rewrite rules such as "β-reduction", "η-reduction".

More precisely, a typed λ-calculus is

1) A set of <u>types</u> s.t.

a) There's a type 1.

b) Given types A and B, there's the product type $A \times B$ & function type $\hom(A,B)$

2) For each type A, a set of <u>terms</u> of that type.
If t is a term of type A we write $t \in A$.
We require:

sometimes written $t : A$

a) For each type A, a countably infinite collection of <u>variables</u> $x_i^A \in A$. (We'll often be lazy and write variables merely as $x, y, z, \ldots$)

b) $\exists \; * \in 1$.

c) If $a \in A$, $b \in B$ are arbitrary terms, there's a term $(a,b) \in A \times B$.

our (a,b) sometimes written $\langle a,b \rangle$

Given a term $c \in A \times B$, there are terms

$$\pi_1^{A,B}(c) \in A \quad \& \quad \pi_2^{A,B}(c) \in B$$

(We'll often be lazy & write these as $\pi_1(c)$ & $\pi_2(c)$.)

d) If $f \in \hom(A,B)$ & $a \in A$ we have a term $f(a) \in B$.

$f(a)$ also called $\varepsilon_f^{A,B}(a)$

e) If x is a variable of type A & $\varphi(x) \in B$ is any term involving x, then we have a term

$$x \in A \longmapsto \varphi(x)$$

of type $\hom(A,B)$

also called $\lambda x : A . \varphi(x)$

3) Relations between terms. Every relation is of the form

$$a \underset{X}{=} a'$$

where a, a' are terms of type A & X is a ~~list~~ finite set of variables which includes all the free variables in a or a'. (A variable x ceases to be free – it becomes bound – when it appears in an expression $x \in A \longmapsto \varphi(x)$. I.e. here x is a dummy variable.)

a) $\underset{X}{=}$ is an equivalence relation (in Lambek & Scott's definition — we'll probably drop the symmetry!)

$$a \underset{X}{=} a$$

$$a \underset{X}{=} a' \;\&\; a' \underset{X}{=} a'' \Longrightarrow a \underset{X}{=} a''$$

$$a \underset{X}{=} a' \Longrightarrow a' \underset{X}{=} a$$

b) If $X \subseteq X'$ then

$$a \underset{X}{=} a' \Longrightarrow a \underset{X'}{=} a'$$

c) If $x \in A$ & $\varphi(x) \underset{X \cup \{x\}}{=} \varphi'(x)$, then

$$(x \in A \longmapsto \varphi(x)) \underset{X}{=} (x \in A \longmapsto \varphi'(x))$$

d) $a \underset{X}{=} a' \Longrightarrow f(a) \underset{X}{=} f(a')$

e) $a \underset{X}{=} *$ $\quad \forall a \in 1$

f) $\pi_1(a,b) \underset{X}{=} a \qquad \pi_2(a,b) \underset{X}{=} b$

$$(\pi_1(c), \pi_2(c)) \underset{X}{=} c \quad \text{for all } c \in A \times B$$

g) $(x \in A \longmapsto \varphi(x))(a) \underset{X}{=} \varphi(a)$

for all $a \in A$ s.t. no free variable in a becomes bound in $\varphi(a)$

h) $x \in A \longmapsto f(x) \underset{X}{=} f$

if $f \in \hom(A,B)$ & $x \in X$ (i.e. a variable in the set X).

i) $x \longmapsto \varphi(x) \underset{X}{=} y \longmapsto \varphi(y)$

if no free occurence of x in $x \longmapsto \varphi(x)$ becomes bound in $y \longmapsto \varphi(y)$.