13 February 2007

An Example of Path Integral Quantization

We have a strategy for quantization given any category C (of "configurations" & "paths") and functor

$$S : C \longrightarrow (\mathbb{R}, +) \qquad \text{(the "action")}$$

This gives a functor

$$e^{iS/\hbar} : C \longrightarrow (\mathbb{C}, \cdot)$$

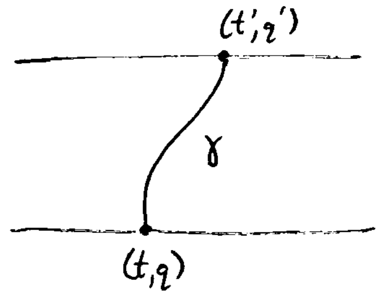
& we compute the "transition amplitude" from any object $x \in C$ to any $y \in C$ via:

$$Z_\hbar(x,y) = \int_{\gamma : x \to y} e^{iS(\gamma)/\hbar} \mathcal{D}\gamma$$

Let's do an example — the free particle on $\mathbb{R}$. Here the objects of C form the set $\mathbb{R}^2 \ni (t,q)$ & morphisms $\gamma : (t,x) \longrightarrow (t',q')$ are paths:

$$\gamma : [t,t'] \longrightarrow \mathbb{R}$$

$$\text{s.t.} \quad \gamma(t) = q \quad \& \quad \gamma(t') = q'$$



So: $$Z_\hbar((t,q),(t',q')) = \int_{\gamma:(t,q)\to(t',q')} e^{iS(\gamma)/\hbar}\, \mathcal{D}\gamma$$

where S is the action for a free particle of mass m:

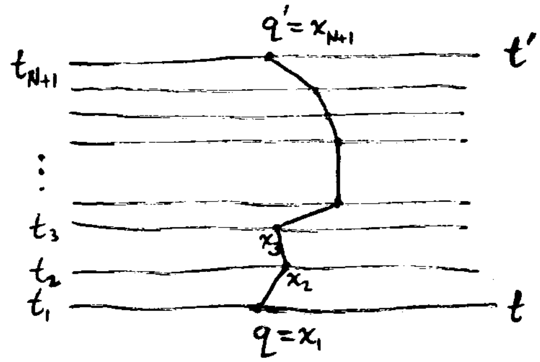
$$S(\gamma) = \int_t^{t'} L(\gamma(s),\dot\gamma(s))\,ds$$

where the Lagrangian is just

$$L(q,\dot q) = \frac{m}{2}\dot q^2$$

since there's no potential.

To do the integral over all paths γ, we first integrate only over piecewise linear paths like:



for some chosen times $t = t_1 < t_2 < \cdots < t_{N+1} = t'$.

To integrate over all these piecewise-linear paths, we just integrate over $x_2, \ldots, x_N$ where $x_i = \gamma(t_i)$. Then we'll try to show that these integrals over piecewise-linear paths converge as the "mesh spacing" $\max_i (t_{i+1} - t_i)$ goes to zero. First, let's see what these integrals look like — let's compute one:

$$A := \int_{\mathbb{R}^{N-1}} e^{\frac{i}{\hbar}\int_t^{t'} \frac{m}{2}\dot\gamma(s)^2 ds} \, dx_2 \, dx_3 \cdots dx_N$$

But γ is piecewise-linear on the ith piece $[t_i, t_{i+1}]$, we have

$$\dot\gamma(s) = \frac{x_{i+1} - x_i}{t_{i+1} - t_i} =: \frac{\Delta x_i}{\Delta t_i}$$

(where $\Delta x_i = x_{i+1} - x_i$, $\Delta t_i = t_{i+1} - t_i$) so we get

$$A = \int_{\mathbb{R}^{N-1}} e^{\frac{i}{\hbar}\frac{m}{2}\sum_{j=1}^{N}\left(\frac{\Delta x_j}{\Delta t_j}\right)^2 \Delta t_j} \, dx_2 \cdots dx_N$$

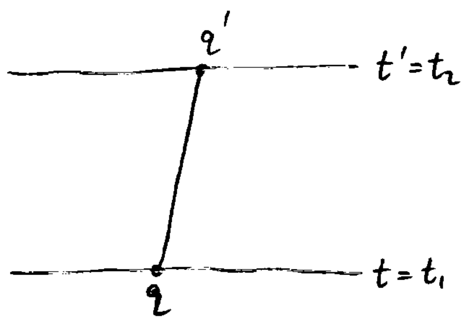
or actually

$$A = \int_{\mathbb{R}^{N-1}} e^{\frac{i}{\hbar}\frac{m}{2}\sum_{j=1}^{N}\left(\frac{\Delta x_j}{\Delta t_j}\right)^2 \Delta t_j} \, \frac{dx_2}{c_2} \cdots \frac{dx_N}{c_N}$$

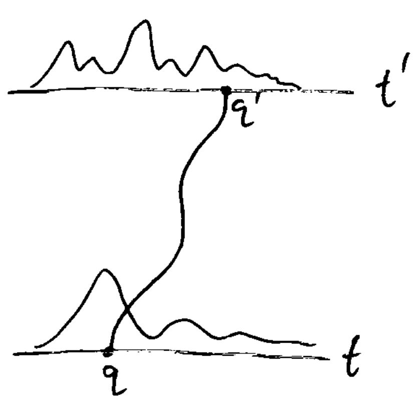
where we rescale Lebesgue measure by a normalizing factor c_i on the ith piece which depends on $t_{i+1} - t_i$. These normalizing factors are needed to get convergence as the mesh spacing goes to zero. But much better: if we pick the c_i correctly, A is actually *independent* of the mesh

$$t = t_1 < t_2 < t_3 < \cdots < t_N < t_{N+1} = t'$$

so convergence is trivial. In other words, we can compute $Z_\hbar((t,q),(t',q'))$ as an integral over *linear* paths:



of which there is just one! To prove that A is independent of the mesh, let's think instead about the rule for evolving a wavefunction ψ in time:



$$\psi(t',q') \in \mathbb{C}$$

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$$\psi(t',q') = \int_{\mathbb{R}} \int_{\gamma:(t,q)\to(t',q')} e^{iS(\gamma)/\hbar}\, \psi(t,q)\, \mathcal{D}\gamma\, dq$$

$$= \int_{\mathbb{R}} Z_\hbar((t,q),(t',q'))\, \psi(t,q)\, dq$$

When we approximate $Z_\hbar$ by integrating over piecewise linear paths, we get

$$\tilde{\psi}(t',q') = \int_{\mathbb{R}^N} e^{\frac{im}{2\hbar}\frac{(\Delta x_N)^2}{\Delta t_N}} \cdots e^{\frac{im}{2\hbar}\frac{(\Delta x_1)^2}{\Delta t_1}}\, \psi(t,q)\, \overset{=dq}{\overset{\downarrow}{dx_1}}\, \frac{dx_2}{c_2} \cdots \frac{dx_N}{c_N}$$

$$= \int_{\mathbb{R}^N} K(\Delta t_N, \Delta x_N) \cdots K(\Delta t_1, \Delta x_1)\, \psi(t,q)\, dq\, dx_2 \cdots dx_N$$

where

$$K(\Delta t_i, \Delta x_i) = \frac{1}{c_i}\, e^{\frac{im}{2\hbar}\frac{(\Delta x_i)^2}{\Delta t_i}}$$

So: to show A is independent of the mesh, we just need to show $\tilde{\psi}(t',q')$ is independent of the mesh for all $\psi(t,-)$.

To show this, it's enough to show:

$$K(t_3-t_1, x_3-x_1) = \int K(t_3-t_2, x_3-x_2) K(t_2-t_1, x_2-x_1)\, dx_2$$

for a suitable choice of c_i.