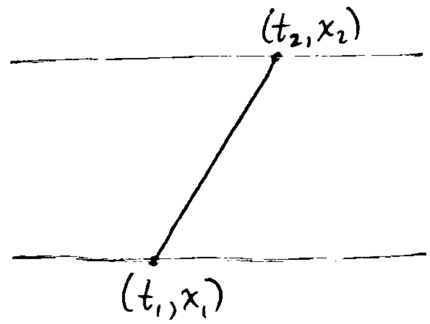
20 Feb 2007

Last time we considered path integral quantization for a free particle on a line and claimed that the exact answer could be found by considering just <u>one</u> path, the straight-line path:



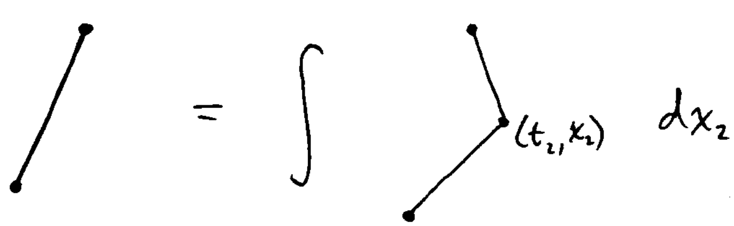
& essentially evaluating $e^{iS/\hbar}$ on this one path, getting this amplitude to go from (t_1, x_1) to (t_2, x_2):

$$K(\Delta t, \Delta x) = \frac{e^{\frac{i}{\hbar}\frac{m(\Delta x)^2}{2\Delta t}}}{c(\Delta t)}$$

where $c(\Delta t)$ is a normalizing factor. We saw that to prove this, we just need to check:

$$(\star) \qquad K(t_3-t_1, x_3-x_1) = \int K(t_3-t_2, x_3-x_2)\, K(t_2-t_1, x_2-x_1)\, dx_2$$

i.e. in pictures



To prove (☆) (for some $c(\Delta t)$) we could just do the integral – but this is too annoying. We'll take a more conceptual route: consider the operator $\mathcal{U}(t)$ which describes one step of time evolution:

$$\left(\mathcal{U}(t_2-t_1)\psi_{t_1}\right)(x_2) = \int K(t_2-t_1, x_2-x_1)\,\psi_{t_1}(x_1)\,dx_1$$

This tells us the wavefunction at time t_2 in terms of the wavefunction ψ_{t_1} at time t_1 as an integral over straight line paths from (t_1,x_1) to (t_2,x_2). In these terms, (☆) says simply

$$\mathcal{U}(t_3-t_1)\psi = \mathcal{U}(t_3-t_2)\,\mathcal{U}(t_2-t_1)\psi$$

i.e.

$$\mathcal{U}(t+s) = \mathcal{U}(t)\,\mathcal{U}(s) \qquad \forall t,s \in \mathbb{R}$$

We would know this if we could write

$$\mathcal{U}(t) = e^{-itH/\hbar}$$

for some operator H, since then we get

$$e^{-i(t+s)H/\hbar} = e^{-itH/\hbar}e^{-isH/\hbar}$$

So, we'll show $U(t) = e^{-itH/\hbar}$ for

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2},$$

the Hamiltonian for the free particle. When we show this, we'll see that if

$$\psi_t = U(t)\psi_0$$

then ψ_t will satisfy Schrödinger's equation (setting $\hbar = 1$)

$$\begin{aligned}\frac{d}{dt}\psi_t &= \frac{d}{dt} e^{-itH}\psi_0 \\ &= -iH e^{-itH}\psi_0 \\ &= -iH\psi_t .\end{aligned}$$

We need to check that $e^{-itH}\psi$ is the same as

$$(U(t)\psi)(x) = \int K(t, x-y)\psi(y)\, dy .$$

Since both of these depend linearly on ψ, it suffices to check the case where ψ is a delta function. Better yet, since both are translation invariant, it suffices to check the case $\psi = \delta$, the Dirac delta at the origin.

Thus we must check that

$$(e^{-itH}\delta)(x) = K(t,x)$$

Note that physically these say the same thing: the left side is the Hamiltonian way of computing the amplitude for a particle to wind up at position x at time t if it starts at the origin at time 0; the right side is the Lagrangian way to compute the same thing — but integrating only over straight-line paths! Since we know $K(t,x)$ (up to the normalizing factor), we'll just compute $(e^{-itH}\delta)(x)$. To do this we'll use the Fourier transform

$$\hat{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x)\, dx$$

and its inverse

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \hat{\psi}(k)\, dk$$

Note:

$$\begin{aligned} \widehat{\frac{d}{dx}\psi}(k) &= \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \frac{d\psi}{dx}\, dx \\ &= ik \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x)\, dx \\ &= ik\, \hat{\psi}(k) \end{aligned}$$

$\int$ by parts

So:

$$\widehat{e^{-itH}\psi}(k) = \widehat{e^{it\frac{1}{2m}\frac{d^2}{dx^2}}\psi}(k)$$

$$= e^{-it\frac{1}{2m}k^2}\hat{\psi}(k)$$

expand exponential as taylor series and pull out

Let's pick units where $\hbar = 1$ & $m = 1$ to lessen the mess.

So we know

$$\widehat{e^{-itH}\delta}(k) = e^{-it\frac{k^2}{2}}\hat{\delta}(k)$$

but

$$\hat{\delta}(k) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx}\delta(x)\,dx = \frac{1}{\sqrt{2\pi}}$$

so

$$\widehat{e^{-itH}\delta}(k) = \frac{1}{\sqrt{2\pi}}e^{-itk^2/2}$$

Now take the inverse Fourier transform:

$$e^{-itH}\delta(x) = \frac{1}{2\pi}\int e^{ikx}e^{-itk^2/2}\,dk$$

$$= \frac{1}{2\pi}\int e^{-\frac{i}{2}(tk^2-2xk+\frac{x^2}{t})+\frac{i}{2}\frac{x^2}{t}}\,dk$$

$$= \frac{1}{2\pi}e^{+\frac{i}{2}\frac{x^2}{t}}\int e^{-\frac{i}{2}(\sqrt{t}k-\frac{1}{\sqrt{t}}x)^2}\,dk$$

$dw = \sqrt{t}\,dk$

$dk = \frac{1}{\sqrt{t}}\,dw$

$$= \frac{1}{2\pi} \frac{e^{+\frac{i}{2}\frac{x^2}{t}}}{\sqrt{t}} \int e^{-\frac{i}{2}u^2}\, du$$

So, just as desired

$$(e^{-itH}\delta)(x) = K(t,x)$$

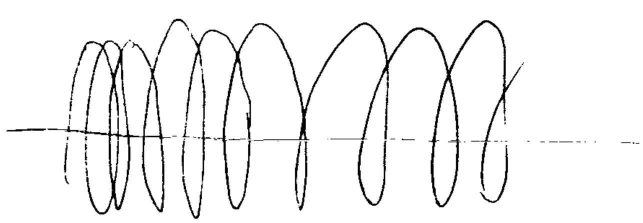
where

$$K(t,x) = \frac{e^{\frac{i}{2}\frac{x^2}{t}}}{c(t)}$$

where the normalizing factor is

$$\frac{1}{c(t)} = \frac{1}{2\pi} \frac{1}{\sqrt{t}} \int e^{-\frac{i}{2}u^2}\, du$$

Note the integral is not absolutely convergent

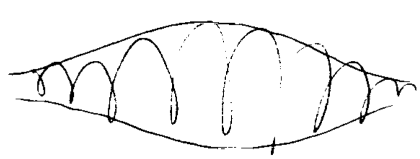


(graph in $\mathbb{R} \times \mathbb{C}$)

but we could consider

$$\int e^{-\frac{k}{2}u^2}\, du$$

where k is close to i but has a small positive real part:



This converges absolutely, and we can take the limit as $k \to i$.

Let's do it:

$$\int e^{-\frac{k}{2}u^2}\,du = \sqrt{\int e^{-\frac{k}{2}x^2}\,dx \int e^{-\frac{k}{2}y^2}\,dy}$$

$$= \sqrt{\iint e^{-\frac{k}{2}(x^2+y^2)}\,dx\,dy}$$

$$= \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-\frac{k}{2}r^2}\,r\,dr\,d\theta} \qquad v = \frac{r^2}{2}$$

$$= \sqrt{2\pi \int_0^{\infty} e^{-kv}\,dv}$$

$$= \sqrt{\frac{2\pi}{k}}$$

So:

$$K(t,x) = \frac{e^{\frac{i}{2}\frac{x^2}{t}}}{2\pi\sqrt{t}} \cdot \sqrt{\frac{2\pi}{i}} = \frac{e^{\frac{i}{2}\frac{x^2}{t}}}{\sqrt{2\pi i t}}$$