

More Examples of Path Integrals

Now let's consider a quantum particle of mass m on the real line, but in a potential

$$V: \mathbb{R} \rightarrow \mathbb{R}$$

Let's compute its time evolution using a path integral, using the fact that we already found the answer for the free particle. Our particle can trace out any path

$$\gamma: [0, T] \rightarrow \mathbb{R}$$

(where w.l.o.g. we start the clock ticking at 0) and its action is

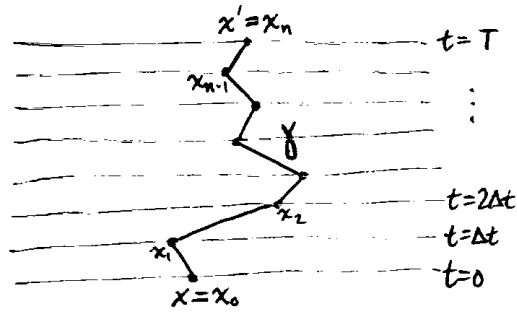
$$S(\gamma) = \int_0^T \frac{m}{2} \dot{\gamma}(t)^2 - V(\gamma(t)) dt$$

so the path integral philosophy tells us given a wavefunction $\psi_0 \in L^2(\mathbb{R})$ at time 0, it will evolve to $\psi_T \in L^2(\mathbb{R})$ at time T where

$$\psi_T(x') = \int_{\substack{\gamma: [0, T] \rightarrow \mathbb{R} \\ \text{with } \gamma(0) = x \\ \gamma(T) = x'}} e^{iS(\gamma)/\hbar} \psi_0(x) \mathcal{D}\gamma dx$$

To do this, we first integrate over piecewise linear paths,

like this:



$$\Delta t = T/n \quad \Delta x_k = x_k - x_{k-1}$$

$$\gamma(k\Delta t) = x_k$$

and then take the limit as $n \rightarrow \infty$, if it exists. So we hope:

$$\psi_T(x') = \lim_{n \rightarrow \infty} \int e^{iS(\gamma)/\hbar} \psi_0(x_0) \frac{dx_0}{c(\Delta t)} \dots \frac{dx_{n-1}}{c(\Delta t)}$$

where

$$\begin{aligned} S(\gamma) &= \int_0^T \frac{m}{2} \dot{\gamma}(t)^2 - V(\gamma(t)) dt \\ &\approx \sum_{k=1}^n \left(\frac{m}{2} \left(\frac{\Delta x_k}{\Delta t} \right)^2 - V(x_{k-1}) \right) \Delta t \end{aligned}$$

So we hope:

$$\psi_T(x') = \lim_{n \rightarrow \infty} \int \prod_{k=1}^n e^{\frac{i}{\hbar} \frac{m}{2} \frac{\Delta x_k^2}{\Delta t}} e^{-\frac{i}{\hbar} V(x_{k-1}) \Delta t} \psi(x_0) \frac{dx_0}{c(\Delta t)} \dots \frac{dx_{n-1}}{c(\Delta t)}$$

In this formula, we're repeatedly multiplying our wavefunction by $e^{-\frac{i}{\hbar} V \Delta t}$ and then evolving it for a time Δt as if it were a free particle. The latter type of evolution, we've seen, amounts to

$$\psi \longmapsto e^{-iH\Delta t/\hbar} \psi$$

where H_0 is the Hamiltonian for a free particle:

$$H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

So

$$\psi_T = \lim_{n \rightarrow \infty} \left(e^{-\frac{i}{\hbar} H_0 \Delta t} e^{-\frac{i}{\hbar} V \Delta t} \right)^n \psi_0.$$

The Lie-Trotter theorem says: if A & B are self-adjoint (unbounded!) operators such that $A+B$ is essentially self adjoint on $D(A) \cap D(B)$, then

$$e^{i(A+B)t} \psi = \lim_{n \rightarrow \infty} \left(e^{iAt/n} e^{iBt/n} \right)^n \psi$$

for all ψ in our Hilbert space (and $\forall t$, $e^{i(A+B)t}$ is unitary). For details*, see vol. 1 of Reed & Simon's *Methods of Modern Mathematical Physics*.

In fact H & V are self-adjoint on $L^2(\mathbb{R})$ & H_0+V is essentially self-adjoint on $D(H_0) \cap D(V)$ if V is reasonably nice (e.g. continuous and bounded below). So in this case

$$\lim_{n \rightarrow \infty} \left(e^{-\frac{i}{\hbar} H_0 \Delta t} e^{-\frac{i}{\hbar} V \Delta t} \right)^n \psi$$

exists and equals

$$e^{-\frac{i}{\hbar} (H_0+V) T} \psi.$$

So we get

$$\psi_T = e^{-\frac{i}{\hbar}(H_0+V)T} \psi_0 = e^{-\frac{i}{\hbar}HT} \psi_0$$

where

$$H = H_0 + V$$

is the Hamiltonian. ~~So we get~~ If $\psi_0 \in D(H_0+V)$

we can differentiate this and get

$$\frac{d}{dt} \psi_t = -\frac{i}{\hbar} (H_0+V) \psi_t$$

which is Schrödinger's equation.

We can also handle the case of a particle on a complete Riemannian manifold, Q . Here again

$$H_0 = -\frac{\hbar^2}{2m} \nabla^2$$

&

" V " = multiplication by $V: Q \rightarrow \mathbb{R}$

are self adjoint operators on $L^2(Q)$ & if V is continuous & bdd below, H_0+V is ess. s.a. on $D(H) \cap D(V)$. So again, skipping lots of steps, we obtain this formula for time evolution of a wavefunction

for a particle on Q with potential V :

$$\begin{aligned}\psi_T &= \lim_{n \rightarrow \infty} \left(e^{-\frac{i}{\hbar} H_0 \frac{T}{n}} e^{-\frac{i}{\hbar} V \frac{T}{n}} \right)^n \psi_0 \\ &= e^{-\frac{i}{\hbar} H T} \psi_0\end{aligned}$$

↙ Ge-Trotter
Formul

where $H = H_0 + V$ is the Hamiltonian for our particle. But, we can't compute $e^{-\frac{i}{\hbar} H T}$ exactly as an integral over piecewise geodesic paths with only a few pieces — need to take a limit as the number of pieces goes to infinity.

Now let's return to our general story where we have a category \mathcal{C} whose objects are thought of as "configurations" and whose morphisms are thought of as "paths". In the example we just saw, objects were points in $\mathbb{R} \times Q$ (spacetime) & a morphism

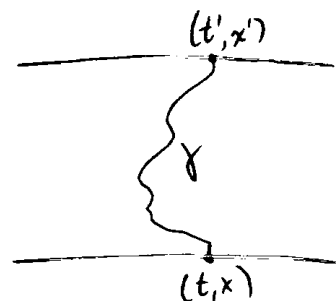
$$\gamma: (t, x) \longrightarrow (t', x')$$

is a path

$$\gamma: [t, t'] \longrightarrow Q$$

s.t.

$$\gamma(t) = x \text{ \& \ } \gamma(t') = x'$$



We also have a functor

$$S: C \longrightarrow \mathbb{R}$$

(where \mathbb{R} is viewed as a category with one object with addition as composition) serving as our action, giving

$$e^{iS/\hbar}: C \longrightarrow U(1) \subseteq \mathbb{C}$$

How do we get a Hilbert space in this more general framework? In the examples above, we could use $L^2(Q)$ — but there's no "Q" in general: Q came from our ability to "slice" the set of objects $\mathbb{R} \times Q$ into slices $\{t = \text{const}\}$ called "Cauchy surfaces" — surfaces on which we can freely specify "initial data" (= "Cauchy data") for our wavefunction.