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We've seen any typed $\lambda$-calculus $P$ (e.g. the "typed $\lambda$-calculus for commutative rings" - which doesn't take full advantage of the features of the $\lambda$-calculus) gives a cartesian closed category $C_P$ where ~~objects of C~~

objects of $C_P$ are types of $P$

morphisms of $C_P$ come from terms of $P$

equations between morphisms of $C_P$ come from equations in $P$.

Then we can look at <u>models</u> of $P$, which are just cartesian closed functors

$$F : C_P \longrightarrow \mathrm{Set}$$

(where Set could be replaced by any other CCC if we wanted).

Last time we considered $P = \lambda\mathrm{Th}(\mathrm{CommRing})$ - the typed $\lambda$-calculus for comm. rings - and asked, "What's a Cartesian Closed functor

$$F : C_{\lambda\mathrm{Th}(\mathrm{CommRing})} \longrightarrow \mathrm{Set} \quad ?"$$

We guessed that the answer to this question is "a commutative ring", i.e. a "model of the typed $\lambda$-calculus for commutative rings". This is true. Let's look at it: what does $F$ give us? We have $R \in C_{\lambda Th(CommRing)}$, so we get

$$F(R) \in Set.$$

Any CCC has a terminal object, $1$, so we get

$$F(1) \cong 1 \in Set$$

where the isomorphism comes from the fact the a CCF preserves terminal objects. Similarly we get

$$F(R \times R) \cong F(R) \times F(R) \in Set$$

since CCF's preserve products, and

$$F(\hom(R,R)) \cong \hom(F(R), F(R)) \in Set$$

since CCF's presere internal homs.

In $C_{\lambda Th(CommRing)}$ we also have morphisms like

$$\begin{aligned}
+ &: R \times R \longrightarrow R \\
0 &: 1 \longrightarrow R \\
- &: R \longrightarrow R \\
\cdot &: R \times R \longrightarrow R \\
1 &: 1 \longrightarrow R
\end{aligned}$$

and these give morphisms in Set:

$$F(+) : \underset{\substack{\cong \\ F(R)\times F(R)}}{F(R\times R)} \longrightarrow F(R)$$

$$F(0) : \underset{\substack{\cong \\ 1}}{F(1)} \longrightarrow F(R)$$

etc. Finally, these functions satisfy the usual commutative ring axioms... so $F(R)$ is a commutative ring!

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Here's a slightly more interesting example:

Example 2: The typed $\lambda$-calculus for "high school calculus" -- how to differentiate polynomials. This typed $\lambda$-calculus $\lambda Th(Calc)$, includes $\lambda Th(CommRing)$ and one more term, "$\frac{d}{dx}$" or "$D$"

$$D \in \hom(\hom(R,R), \hom(R,R))$$

since $\frac{d}{dx}$ "eats a function and spits out a function". We also have extra equations. For example, we need an equation corresponding to

$$\text{"}\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\text{"}.$$

Formally, we'll say $x$ is a variable of type $R$ and $f$ & $g$ are variables of type $\hom(R,R)$:

$$D\big(\underset{\overset{\cap}{R}}{x} \longmapsto +(f(x),g(x))\big) \underset{\{x,f,g\}}{=\!=} \underset{\overset{\cap}{R}}{x} \longmapsto +\big((D(f))(x),\,(D(g))(x)\big)$$

Similarly we've got equations corresponding to

- product rule
- rule for differentiating "$-f$"
- rule for "$\frac{d}{dx}x = 1$":

$$D(x\in R \longmapsto x) = x\in R \longmapsto 1(*)$$

- rules "$\frac{d}{dx}0 = 0$" "$\frac{d}{dx}1 = 0$".
- chain rule: "$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$"

$$D(x\in R \mapsto f(g(x))) \underset{\{x,f,g\}}{=\!=} x\in R \mapsto \cdot\big((Df)(g(x)),(Dg)(x)\big)$$

If we use this as our (rough) definition of $\lambda Th(Calc)$, what's a model of it like?

$$F: C_{\lambda Th(Calc)} \longrightarrow Set\ ?$$

You might try $F(R) = \mathbb{R} \in Set$, but

$$\begin{aligned} \hom(F(R),F(R)) &\cong \hom(\mathbb{R},\mathbb{R}) \\ &= \{\underline{\underline{\text{all}}} \text{ functions } f:\mathbb{R}\to\mathbb{R}\}. \end{aligned}$$

and we have

$$D: \hom(R,R) \longrightarrow \hom(R,R) \quad \text{in } C_{\lambda Th(Calc)}$$

& thus

$$\begin{array}{ccc} F(D): F(\hom(R,R)) & \longrightarrow & F(\hom(R,R)) \quad \text{in Set.} \\ \cong & & \cong \\ \hom(\mathbb{R},\mathbb{R}) & \longrightarrow & \hom(\mathbb{R},\mathbb{R}) \end{array}$$

Challenge: show no model of this type (i.e. one with $F(R)=\mathbb{R}$) exists, or construct one! (So, perhaps $\lambda Th(Calc)$ describes the "freshman's paradise" where all functions are differentiable!)