Hilbert Spaces & Operator Algebras from Categories

Suppose we have a category C of "configurations & processes" and an "action" functor

$$S : C \longrightarrow (\mathbb{R}, +)$$

giving

$$e^{iS} : C \longrightarrow (U(1), \cdot)$$

describing the amplitude for any process to occur. How do we get a Hilbert space out of this? Here's one avenue of attack: First, as a 0th approximation to our Hilbert space, form a vector space as follows. Let $Ob(C)$ be the set of all objects of C & $Mor(C)$ be the set of all morphisms in C. We have the source and target maps

$$s, t : Mor(C) \longrightarrow Ob(C)$$

assigning to each morphism $\gamma : x \to y$ its source $s(\gamma) = x$ and target $s(\gamma) = y$. Form the vector space $Fun(Ob(C))$ of "nice" complex functions on $Ob(C)$, where we'll have to see what sort of niceness we need.

Then, define for $\psi, \phi \in \mathrm{Fun}(\mathrm{Ob}(C))$ an "inner product":

$$\langle \phi, \psi \rangle = \int_{\gamma: x \to y} e^{iS(\gamma)} \bar{\phi}(y) \psi(x) \, \mathcal{D}\gamma \, \mathcal{D}x \, \mathcal{D}y$$

$$= \int_{\mathrm{Mor}(C)} e^{iS(\gamma)} \bar{\phi}(t(\gamma)) \psi(s(\gamma)) \, \mathcal{D}\gamma$$

For this to make sense we really need a measure on $\mathrm{Mor}(C)$ & ψ, ϕ should be nice enough so that the integral converges — e.g. $\phi \circ t, \psi \circ s \in L^2(\mathrm{Mor}(C))$

Now we have questions:

1) Is $\langle \phi, \psi \rangle$ linear in ψ and conjugate linear in ϕ?

2) Is $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$?

3) Is $\langle -, - \rangle$ nondegenerate? (Given ϕ s.t. $\langle \phi, \psi \rangle = 0 \ \forall \psi$, is $\phi = 0$?)

4) Is $\langle \psi, \psi \rangle \geq 0$?

Consider each of these in turn

1) is obvious if the integral is well behaved.

2) is more interesting:

$$\overline{\langle \phi, \psi \rangle} \stackrel{?}{=} \langle \psi, \phi \rangle$$

$$\| \qquad\qquad\qquad \|$$

$$\int_{\gamma : x \to y} e^{-iS(\gamma)} \overline{\psi}(x) \phi(y) \mathcal{D}\gamma \mathcal{D}x \mathcal{D}y \qquad \int_{\gamma : y \to x} e^{iS(\gamma)} \overline{\psi}(x) \phi(y) \mathcal{D}\gamma \mathcal{D}x \mathcal{D}y$$

This is related to time reversal symmetry. It's almost immediate if C is a groupoid, since then given

$$\gamma : x \longrightarrow y$$

we get

$$\gamma^{-1} : y \longrightarrow x$$

and since S is a functor

$$S(\gamma^{-1}) = -S(\gamma)$$

so we'll get $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$ <u>if</u> the measure $\mathcal{D}\gamma$ on $\mathrm{Mor}(C)$ is preserved by the transformation

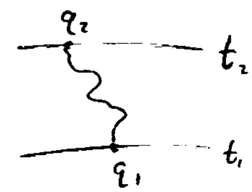
$$^{-1} : \mathrm{Mor}(C) \longrightarrow \mathrm{Mor}(C).$$

<u>But</u> our favorite example is not a groupoid! Recall given a manifold Q we have a category w.

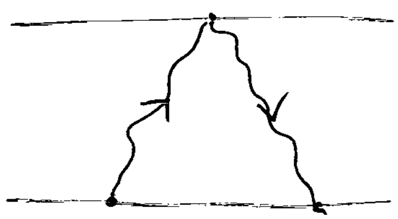
$$\mathrm{Ob}(C) = \mathbb{R} \times Q$$

where a morphism $\gamma : (t_1, q_1) \to (t_2, q_2)$ is a path

$$\gamma : [t_1, t_2] \longrightarrow Q \quad \text{s.t. } \gamma(t_i) = q_i$$



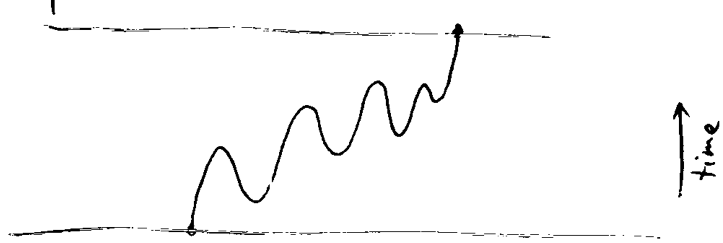
Here we've been assuming $t_1 \leq t_2$, so this is not a groupoid. We <u>could</u> adjoin inverses formally to get a groupoid, but then we'd get morphisms like:



which do indeed show up in Feynman diagrams involving antimatter, but would require further thought.

Research topics:

I) Study Feynman's original work on path integrals for a special relativistic particle and see if he allowed paths like this:



II) If so, formalize what he did using some category C. Is it a groupoid or merely a <u>$*$-category</u>?

<u>Def</u>: A <u>$*$-category</u> is a category C with a contravariant functor $* : C \longrightarrow C$ that's the identity on objects & has $** = 1_C$.
Equivalently, for any morphism $\gamma : x \longrightarrow y$ there's a morphism $\gamma^* : y \longrightarrow x$ s.t.

1) $(\gamma_1 \circ \gamma_2)^* = \gamma_2^* \circ \gamma_1^*$
2) $(\gamma^*)^* = \gamma$.

(these imply $1_x^* = 1_x \quad \forall x \in C$)

This is sometimes called a "~~categorry~~ category with involution", or in quantum computer science, a "$\dagger$-category".

The main example is Hilb, the category of Hilbert spaces & bounded linear operators: given $T : H \to H'$ we get $T^* : H' \longrightarrow H$ defined by

$$\langle T^* \phi, \psi \rangle = \langle \phi, T\psi \rangle \qquad \forall \psi \in H \ \phi \in H'$$

3) $\langle -,- \rangle$ is usually degenerate, but that's OK: we can form

$$K \subseteq \mathrm{Fun}(\mathrm{Ob}(C))$$

by

$$K = \{ \psi : \langle \phi, \psi \rangle = 0 \quad \forall \phi \in \mathrm{Fun}(\mathrm{Ob}(C)) \}$$

and form the quotient space

$$H_0 = \mathrm{Fun}(\mathrm{Ob}(C))/K$$

on which we have $\langle -,- \rangle$ defined by

$$\langle [\phi], [\psi] \rangle := \langle \phi, \psi \rangle$$

and this is nondegenerate.

4) Is $\langle \psi, \psi \rangle \geq 0$? To get this, we need some extra conditions... but we'd need to look at some examples to find <u>nice</u> sufficient conditions. This is somehow related to "reflection positivity" in the Osterwalder-Schrader Theorem. If we get $\langle \psi, \psi \rangle \geq 0$, we can complete H_0 & get a Hilbert space.

Besides the issue of Hilbert spaces, there's the issue of operators. How can we get some nice operators on $Fun(Ob(C))$? We can get them from elements $F \in Fun(Mor(C))$, some space of "nice" functions on $Mor(C)$:

$$(F\psi)(y) := \int_{\gamma: x \to y} F(\gamma)\psi(x)\, \mathcal{D}\gamma\, \mathcal{D}x$$

where "nice" means this converges. In fact we get an algebra of such operators, with some luck:

$$((GF)(\gamma) = \int_{\substack{\gamma_1, \gamma_2 \\ \text{s.t. } \gamma_2 \circ \gamma_1 = \gamma}} G(\gamma_2) F(\gamma_1)\, \mathcal{D}\gamma_1 \mathcal{D}\gamma_2$$

This is "convolution"; $Fun(Mor(C))$ is called the "category algebra" of C.