The Big Picture

Last time we sketched how to get a Hilbert space from a category C (of "configurations" & "processes") equipped with an "amplitude" functor

$$A : C \longrightarrow U(1)$$

There are lots of subtleties involving analysis, but these evaporate when C is finite -- let's consider this case for simplicity. Then we form a vector space $\mathrm{Fun}(\mathrm{Ob}(C))$ — which now means <u>all</u> functions

$$\psi : \mathrm{Ob}(C) \longrightarrow \mathbb{C}.$$

$\mathrm{Fun}(\mathrm{Ob}(C))$ is isomorphic to $\mathbb{C}[\mathrm{Ob}(C)]$ — the space of formal linear combinations of objects of C. Then we define a sesquilinear map

$$\langle -, - \rangle : \mathbb{C}[\mathrm{Ob}(C)] \times \mathbb{C}[\mathrm{Ob}(C)] \longrightarrow \mathbb{C}$$

linear in this argument

conjugate linear in this argument

by
$$\langle y, x \rangle = \sum_{\gamma : x \to y} A(\gamma) \qquad x, y \in \mathrm{Ob}(C)$$

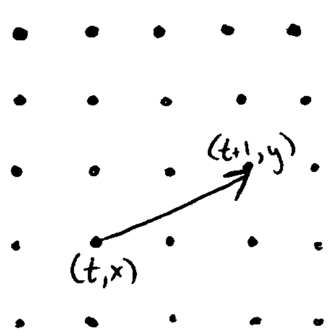
(and extending to $\mathbb{C}[\mathrm{Ob}(C)]$ by linearity.)

We're doing a "path integral", but now it's a mere sum over morphisms — we're implicitly using counting measure on $\mathrm{Hom}(x,y)$. This inner product may be degenerate. Take $\mathbb{C}[\mathrm{Ob}(C)]$ & mod out by

$$\{\psi \in \mathbb{C}[\mathrm{Ob}(C)] : \langle \psi, \phi \rangle = 0 \ \forall \phi \in \mathbb{C}[\mathrm{Ob}(C)]\}$$

to get a vector space H with nondegenerate sesquilinear form on it; if that's positive definite, then H is a Hilbert space.

Alex Hoffnung and J.B. have been considering examples like this: "a particle on a (discretized) line":



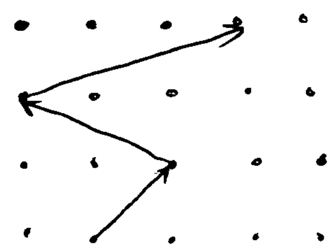
$$\mathrm{Ob}(C) = \{1, \dots, T\} \times \{1, \dots, X\}$$

Morphisms in C are freely generated by morphisms

$$\gamma : (t,x) \longrightarrow (t+1,y)$$

$$\forall t \in \{1, \dots, T-1\} \quad \forall x,y \in \{1, \dots, X\}$$

So a typical morphism in C looks like

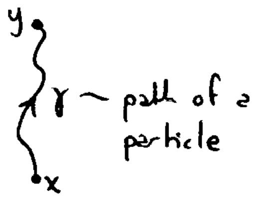


(A category freely generated in this way is called a "quiver.") If you choose the amplitude $A: C \to U(1)$ to be a discretized version of the amplitude for a particle on a line, we recover standard physics in the continuum limit.

We really want to categorify all this...

PARTICLES

A category C



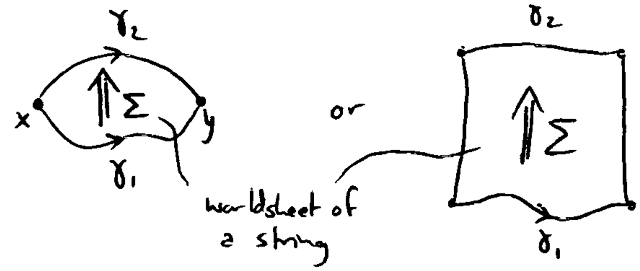
A functor

$$A : C \longrightarrow U(1) \subseteq \mathbb{C}$$

the category with
one object $*$
elts of $U(1)$ as morphisms

STRINGS

A 2-category or a double category



A 2-functor

$$A : C \longrightarrow U(1)[1]$$

"shifted $U(1)$"

the 2-category with
one object $*$
one morphism 1_*
elts of $U(1)$ as 2-morphisms

For any abelian group A & any $n \geq 0$, we can form an n-category $A[n]$ — the "n times shifted" version of A

$$U(1)[1] \simeq U(1)\mathrm{Tor}$$

the monoidal category of $U(1)$-torsors.
For any group G a G-set that's isomorphic to G, where G acts on itself by left multiplication. "A torsor is a group that's forgotten its identity".

PARTICLES	STRINGS

If G is abelian, G-Tor is a monoidal category with

$$X \otimes Y = \frac{X \times Y}{(xg, y) \sim (x, gy)}$$

where $X, Y \in G$-Tor & $g \in G$ acts on the right on X since G acts on the left and G is abelian.

- $U(1)$Tor has
  - one object, $*$
  - $U(1)$-torsors as morphisms, with $\otimes$ as composition
  - $U(1)$-torsor morphisms as 2-morphisms.

$U(1)[1]$ is a skeleton of $U(1)$Tor, so in particular $U(1)[1] \simeq U(1)$Tor.

Particles: From $A : C \longrightarrow U(1)$ we try to build a Hilbert space, but first we form the vector space $\mathrm{Fun}(\mathrm{Ob}(C))$, which if C is finite is just

$$\mathrm{Hom}(\mathrm{Ob}(C), \mathbb{C}) \cong \mathbb{C}[\mathrm{Ob}(C)]$$

Strings: From $A : C \longrightarrow U(1)$Tor we try to build a 2Hilbert space, but first we form the 2-vector space $\mathrm{FUN}(\mathrm{OB}(C))$ which if C is finite is just:

$$\mathrm{Hom}(\mathrm{OB}(C), \mathrm{Vect}_{\mathbb{C}}) \overset{?}{\simeq} \mathrm{Vect}_{\mathbb{C}}[\mathrm{OB}(C)].$$

Here $OB(C)$ could be the category formed by discarding the 2-morphisms in our 2-category C — but this only works if C is strict. What should it be in general? Good question.

We define $\langle -,- \rangle$ on $\mathbb{C}[Ob(C)]$ by:

$$\langle y, x \rangle = \sum_{\gamma : x \to y} A(\gamma) \in \mathbb{C}$$

Here we use

$$U(1) \hookrightarrow \mathbb{C}$$

to add elts of $U(1)$ and get elts of $\mathbb{C}$.

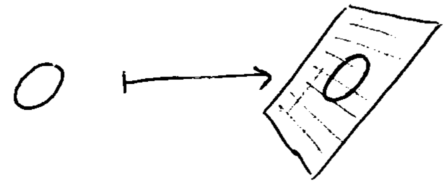
$\langle -,- \rangle$ on $\mathrm{Vect}[OB(C)]$ should satisfy

$$\langle y, x \rangle = \bigoplus_{\gamma : x \to y} A(\gamma) \in \mathrm{Vect}_{\mathbb{C}}$$

Here we use

$$U(1)\mathrm{Tors} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

sending $U(1)$ torsors to their corresponding 1-dimensional vector spaces



— in fact this is a Hilbert space.

For more, see Daniel Freed's "Higher algebraic structures & quantization."