2-Categories from Typed λ-Calculi, cont.

Last time we listed 6 rewrite rules which Lambek & Scott used to create a poset of terms of any given type in any typed λ-calculus, P. They prove (something stronger than this version of) the

<u>Church-Rosser</u> <u>Theorem</u>: Given any set of types, there's ~~the~~ the initial typed λ-calculus P_0 with those types (& nothing else other than what the definition requires). The corresponding 2-category $\tilde{C}_{P_0}$ with:

- types as objects
- $(x \in X, \varphi(x))$ (with $\varphi(x)$ a term of type A) as morphisms $f: X \to A$
- rewrite rules $\varphi \Rightarrow \psi$ generating 2-morphisms.
- all possible equations between 2-morphisms

has the property that for any objects (types) X & Y, the category $\hom(X,Y)$ is a poset, and <u>this poset is terminating & confluent</u>.

Note: Lambek & Scott's 2-category $\tilde{C}_P$ is just a 2-poset, i.e. a 2-category s.t. the hom categories are all posets. We could also create a more interesting 2-category $\tilde{C}_P$ by letting rewrite rules freely generate the hom categories $\mathrm{Hom}(X,Y)$. J.B. believes the Church-Rosser Thm. would still hold, i.e. $\mathrm{Hom}(X,Y)$ would still be terminating and confluent.

Last time we drew surface diagrams for all but two of Lambek & Scott's rewrite rules:

5) β-reduction $\quad (x \in X \longmapsto \varphi(x))(c) \Longrightarrow \varphi(c) \quad$ $\varphi(x) \in A$, $c \in X$

6) η-reduction $\quad (x \in X \longmapsto f(x)) \Longrightarrow f \quad$ $f \in \mathrm{hom}(X,A)$

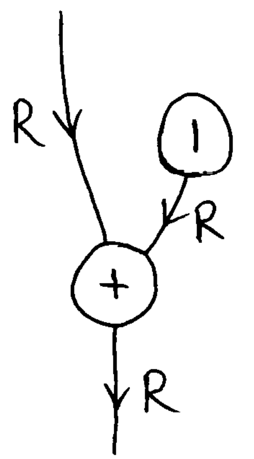
Let's consider (5) in the example where $P = \lambda\mathrm{Th}(\mathrm{Calc})$. For example:

$$(x \in R \longmapsto x+1)(y) \Longrightarrow y+1$$

Terms give us morphisms, e.g.

$$(x \in R,\ x+1)$$

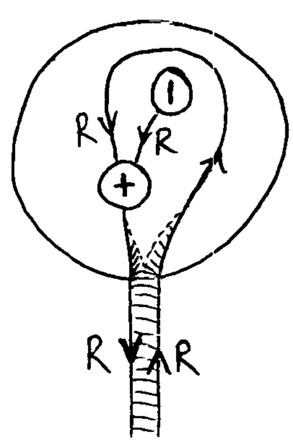
gives a morphism from R to R in $\tilde{C}_{\lambda Th(Calc)}$.
As a string diagram, this morphism looks like:



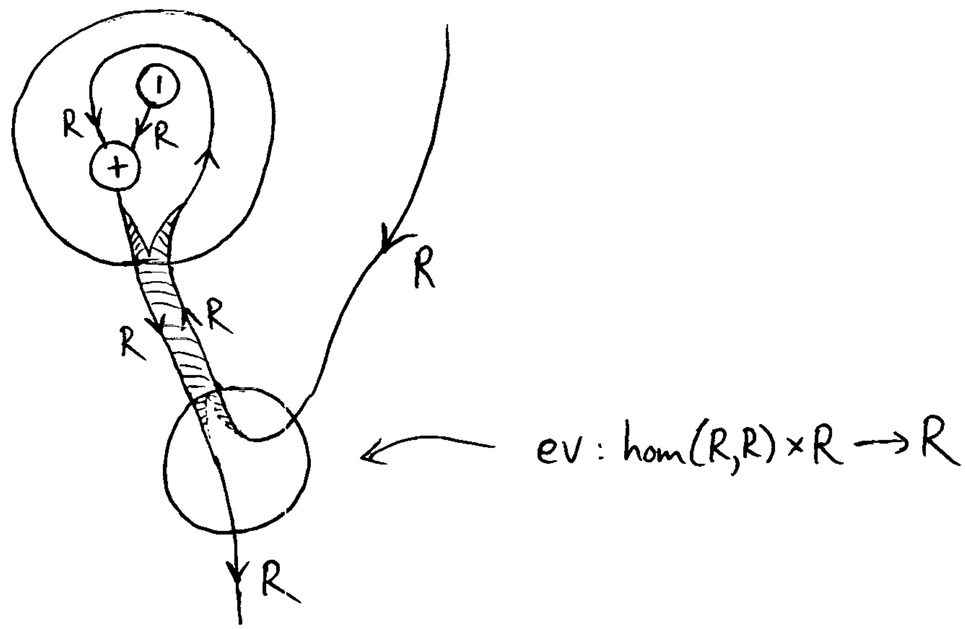
$1 : 1 \longrightarrow R$

$+ : R \times R \longrightarrow R$

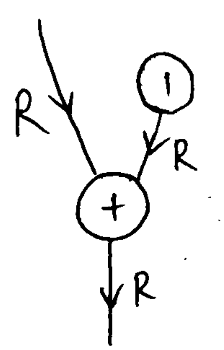
Similarly, $(z \in 1,\ x \in R \mapsto x+1)$ gives a morphism from 1 to $\hom(R,R)$ — this is just the curried version of the previous morphism from R to R. As a string diagram this looks like:



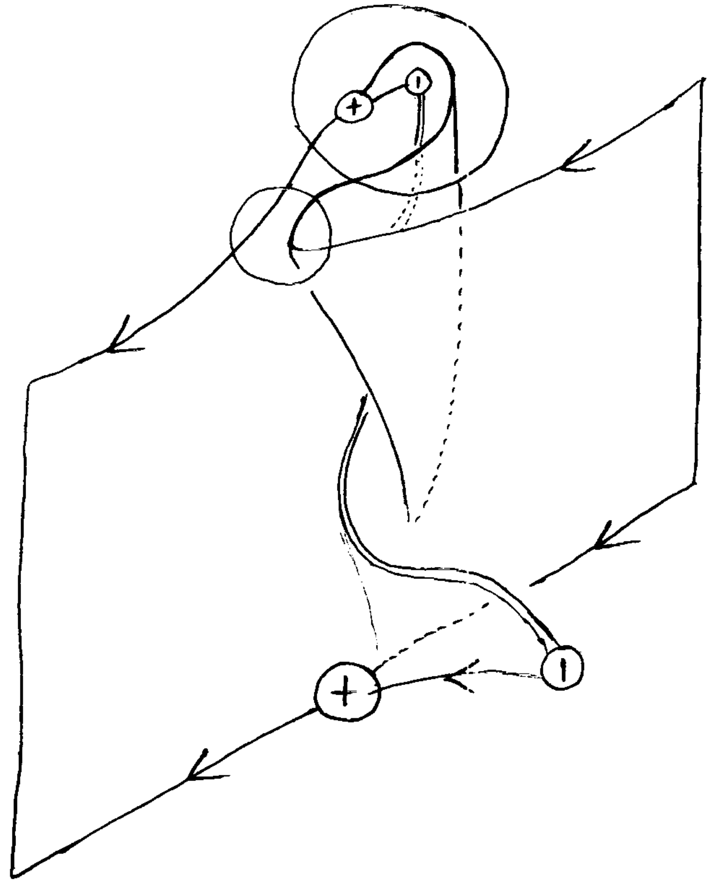
Next, $(y \in R, (x \mapsto x+1)(y))$ gives a morphism from R to R, which has string diagram:



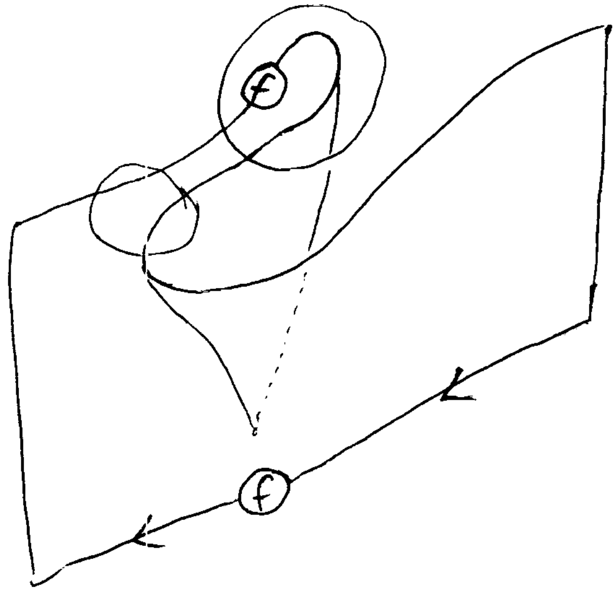
Our rewrite rule 5 — β-reduction — simplifies this to $(y \in R, y+1)$ which we've seen the diagram for before:



So, β-reduction (in this example) can be drawn as a surface diagram:

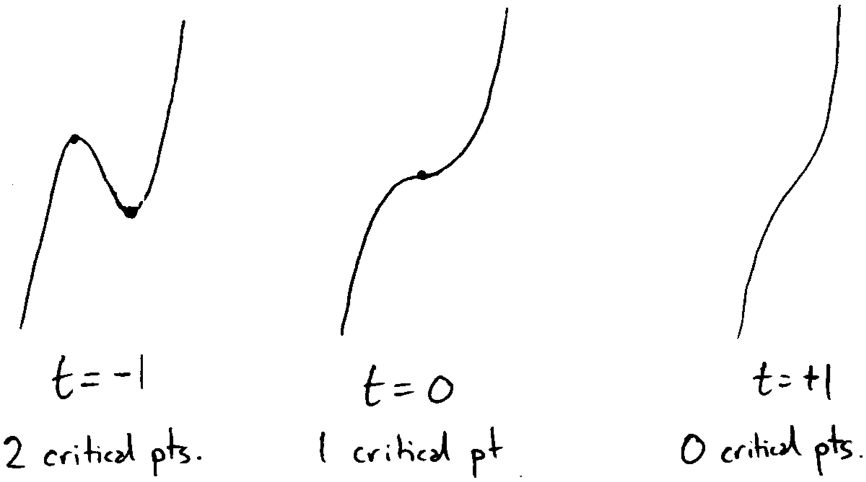


In general, β-reduction looks like this:

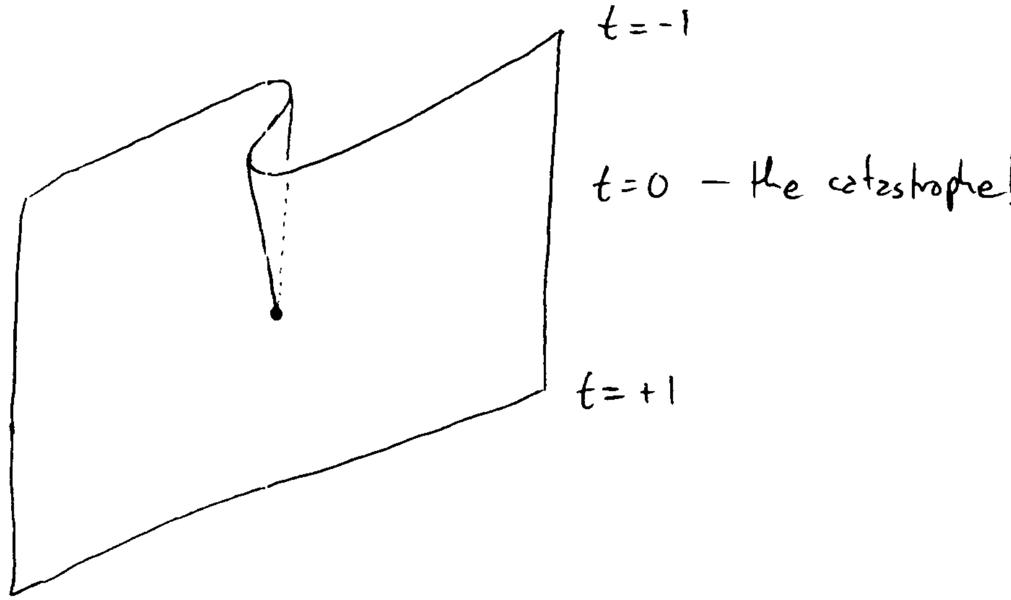


We see here Thom's "fold catastrophe":

$f_t(x) = x^3 + tx$



If we draw "time" t in the vertical direction we get



Thom has a philosophy in which all verbs (processes) correspond to different catastrophes, the fold being the simplest. So: we've seen that "evaluation of a function" (β-reduction) corresponds to the simplest catastrophe.

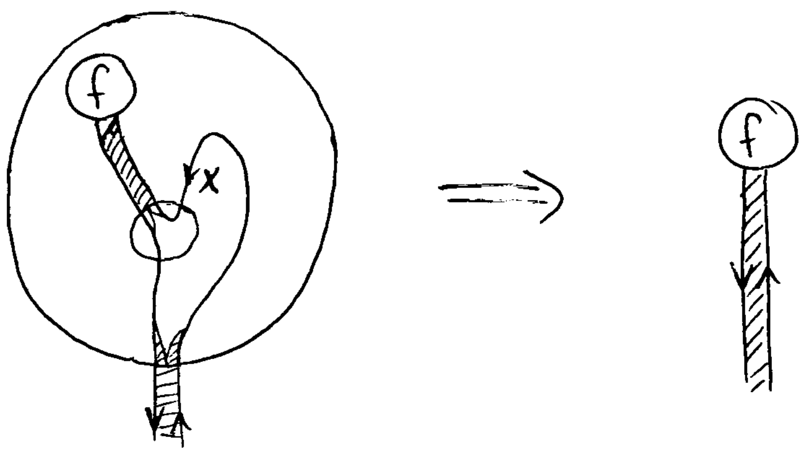
What about η-reduction?

6) $\quad x \in X \longmapsto f(x) \implies f \qquad f \in \hom(X, A)$

This gives a 2-morphism in $\widetilde{C}_P$:

$$(y \in 1,\ x \in X \longmapsto f(x)) \Rightarrow (y \in 1, f)$$

Not quite sure how to draw this, but maybe:



Challenge: draw this as a surface. What catastrophe do you get?