8 | JAN | 2008 | TUE

Geometric Representation Theory – A nonstandard approach

(1) Traditionally, $\downarrow$ = GRT uses Lie groups (gps that also are manifolds)

These are "nice" because they combine geometry (manifold) with symmetry (group).

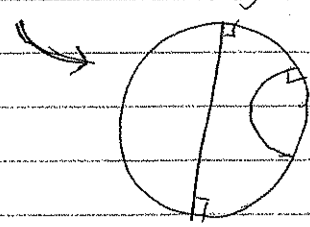
(a) eg: Euclidean geometry $(\mathbb{R}^n, \|\cdot\| = \text{metric})$

The Euclidean group acts as symmetries (i.e. preserving, so isometries).

$$\underbrace{O(n)}_{\text{rot}^n/\text{refl}^n} \ltimes \underbrace{\mathbb{R}^n}_{\text{transl}^n}$$

(b) Non Euclidean geometries eg. $(S^n, \text{usual distance}) \longrightarrow O(n+1)$ (Symm gp)

$(H^n, \text{usual distance}) \longrightarrow O(n,1)$



(c) ~~Bt~~ Then people tried to classify all Lie groups, but this has proved to be too complicated a task as yet. So, the next best task was to look at simple Lie groups

These are ~~Lie~~ gps, whose only normal subgroups are discrete

e.g. $O(n) \longrightarrow UU^* = 1$ over $\mathbb{R}$
$U(n) \longrightarrow UU^* = 1$ over $\mathbb{C}$
$Sp(n) \longrightarrow UU^* = 1$ over $\mathbb{H}$ = quaternions.

And if we want ~~to~~ classify compact Lie gps (and forget about modding out by discrete subgps); then there are only five more "exceptional" Lie groups !

(compact, simple)

These are related to octonions.

ⓓ Next, Lie groups are related to geometries that they are the symmetries of. These are understood; in fact, you can completely understand the non-compact simple Lie groups too - and their geometries.

This is how GRT is understood by most people - but not here!

—— X —— X —— X ——

② Now to continue from where we left off last quarter.

ⓐ The Quantum Harmonic Oscillator - (Mathematically) (QHO)

Has been used to understanding physics - ~~e.g.~~ through QFT's.

But essentially, it's the algebra $\mathbb{C}[z_1, \ldots, z_n]$

There are 2 ways of gen'zing:

(i) <u>Groupoidifying</u>: We get FinSet^n_0 (X^n = Prod. of Categories
$-_0$ = groupoid
FINSET = fin sets)

So, n fin sets + bij^n preserving each of them

This is the same as $[n\text{-colored fin sets}]_0$

(together with an "ordering" of the colors?)

However, this is not easy to generalize in the "other" direction, as we'll say below. So, here's the other way to describe this:

$[\text{fin sets } S \text{ with an } n\text{-stage flag } \emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S]_0$

(ii) <u>q-deforming</u>: q-deformed QHO: $\mathbb{C}_q[z_1,\dots,z_n]$

(b) <u>Question</u>: Can we do both?

$$\begin{tikzcd}
\text{QHO} \; \mathbb{C}[z_1,\dots,z_n] \arrow[rr, "q"] \arrow[d, shift left, "\text{Groupoidify}"] & & q\text{-QHO} \; \mathbb{C}_q[z_1,\dots,z_n] \arrow[d, "\text{Groupoidify}"] \\
{[\text{fin-sets w/ } n\text{-flag}]_0} \arrow[u, shift left, "\text{Degroupoidify}"] \arrow[rr, "q"] & & ??
\end{tikzcd}$$

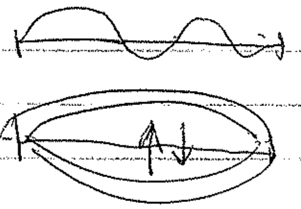
where JB thinks ?? $= [\text{fin dim } \mathbb{F}_q\text{-vect spcs w/ } n\text{-stage flags}]_0$

and we'll try to do this ~~~~ this quarter.

③ The Harmonic Oscillator in Physics

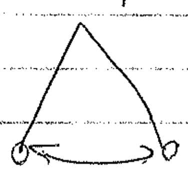
ⓐ One problem in physics was about the particle vs wave theories for light — starting from Newton & Huygens.

The theory that is now used involves vibrational modes of light — but they're not any energy levels, as Maxwell's equations would predict —



but they are quantized !!

ⓑ But this theory works for any harmonic oscillator! eg the pendulum



In nice units, $H = \frac{1}{2}(p^2 + q^2)$

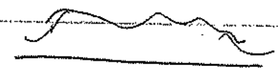
q → pos^n

p → momentum

H → energy / Hamiltonian

$\boxed{n=1}$

(c) So here, p & q are "scalars". But in quantum mechanics, q & p become operators on a Hilbert space $L^2(\mathbb{R})$.



Here, instead of describing states, we have ψ = wavefunction $: \mathbb{R} \to \mathbb{C}$,

~~and~~ so that $\forall$ measurable $U \subseteq \mathbb{R}$,

$$\int_U |\psi|^2 \, dx = \text{prob}(\text{posn} \in U \subseteq \mathbb{R})$$

(d) The operators here are: $(q\psi)(x) := x\psi(x)$

$$(p\psi)(x) := \frac{1}{i}\frac{d\psi}{dx}$$

(should be $\hbar$)

But now p & q don't commute; instead, $\boxed{pq - qp = \frac{1}{i}}$

(e) Now the Hamiltonian becomes an operator $H = \frac{1}{2}(p^2+q^2) : L^2(\mathbb{R})$? with eigenvectors, having eigenvalues $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

~~FACT: These are all the eigenvectors of~~

$\boxed{\text{Homework}}$ (a) $\psi_0(x) = e^{-x^2/2}$ satisfies $H\psi_0 = \frac{1}{2}\psi_0$

(b) H is self-adjoint on $L^2(\mathbb{R})$

(c) Define the creation & annihilation operators by

$$a^* := \frac{1}{\sqrt{2}}(p + iq) \quad , \quad a := \frac{1}{\sqrt{2}}(p - iq)$$

Then (i) a^* & a are adjoint operators

(ii) $[a, a^*] = 1$

(iii) $[H, a^*] = a^*$

$[H, a] = -a$

(iv) $H = a^* a + \frac{1}{2}$

(d) Now show: $H\psi = \lambda\psi \Rightarrow H a^* \psi = (\lambda+1)\psi$

$H a \psi = (\lambda - 1)\psi$.

$\boxed{\text{NOTE}}$ The operators p & q are NOT : $L^2(\mathbb{R})$?

but are defined on the span of $[\text{polynom}\cdot(e^{-x^2/2})]$, $\mathcal{S}$

which is closed under p & q, and $\subseteq L^2(\mathbb{R})$.

But this is fine.

$\boxed{\text{Remark}}$: The homework becomes simpler ~~to do~~ if one starts by showing (via integration by parts) that both p and q are self-adjoint on $\mathcal{S}$.