

JAN/2008/11

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~~$\frac{1}{2} (X/A_0 + Y_0)$~~

① @ Recall, from last quarter, that de-groupoidification takes a groupoid to its zeroth homology.

$$X \mapsto H_0(X) = \text{Free}(\text{components of } X)$$

It also takes spans between groupoids to linear operators.

② We'll consider an example. Let A be any abelian category; so we consider A_0 .

Of course, these are closed under $\oplus = \amalg = \coprod$, so they are never finite unless $\mathcal{C} = \{0\}$.

Essentially, all small abelian categories are $\cong R\text{-Mod}$.

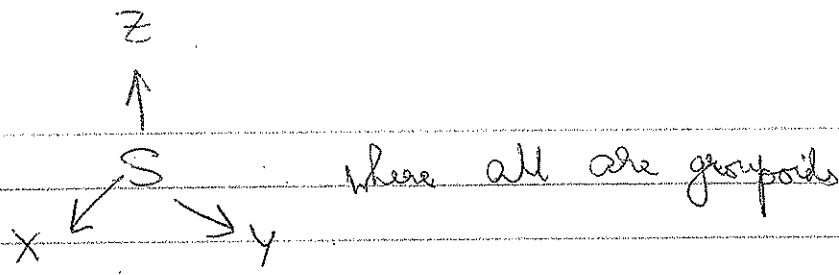
And the name comes from the prototype $\mathbb{Z}\text{-Mod} = \text{category of abelian groups}$.

But we'll try to not get "too infinite" eg. think of the following example:

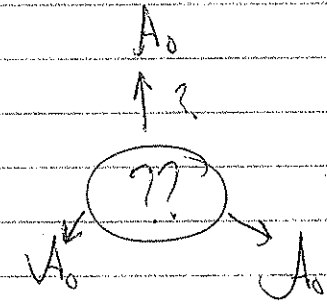
$\mathcal{A} \cong$ fin. dim. reps of a finite quiver over a finite field

③ The idea is to try to get linear operators: $V \times V \rightarrow V$
or equivalently, $V \otimes V \rightarrow V$

And this will come from a tai-span, \square



More precisely, we will de-groupify

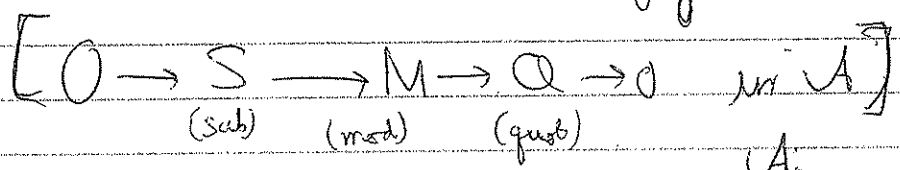


And get a "3-index tensor"

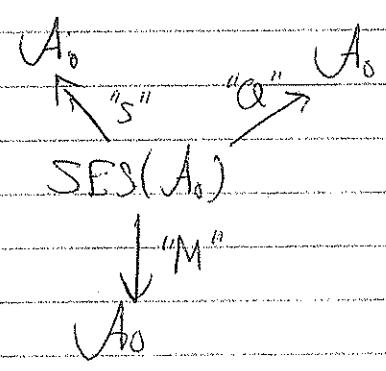
which can be thought of/re-interpreted
 as a bilinear map: $V \otimes V \rightarrow V$
 ($V = H_0(U_0)$?)

(d) In fact, $\textcircled{??} = \text{SES}(A_0)$

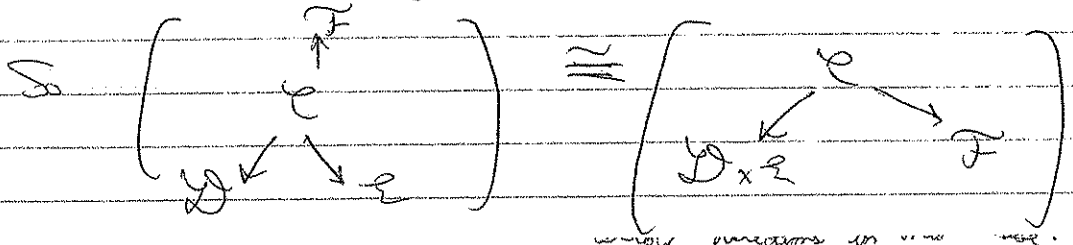
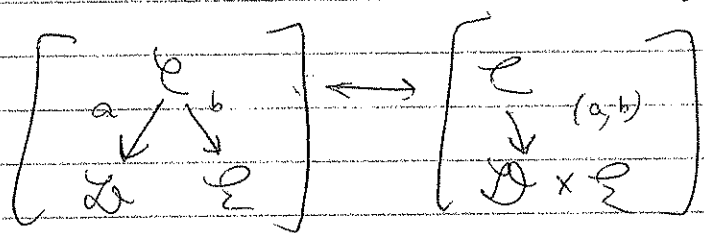
This is a category, and we have elts =



Our tri-span now becomes



Now,



where \mathcal{F} is a map

and by our H_0 -Construction, this gives a linear map

$$: H_0(D \times E) \rightarrow H_0(F)$$

and $\downarrow \cong H_0(D) \otimes H_0(E)$.

(e) So, UPSHOT we seek, from $SES \rightleftharpoons A_0$,

a linear map $: H_0(A_0) \otimes H_0(U_0)$
 $\downarrow m$
 $H_0(U_0)$

AND \rightarrow hope that this map m is "assoc", which would make $H_0(A_0)$ into an asso. alg.

In fact, this does turn out to be the case, and $H_0(A_0)$ is called the Hall algebra of the abelian category \mathcal{A} .

(f) Here's a "vague" reason why we expect the mult. to be associative.

"Mult" three things is like a module of "length" 3,
 is a module with a 3-stage filtration

$$0 \subseteq M'' \subseteq M' \subseteq M$$

$$\text{or } M_0 \subseteq_a M_1 \subseteq_b M_2 \subseteq_c M_3$$

But now assoc is $(ab)c = a(bc)$
 $(M_0 \subseteq_a M_1) \subseteq_b M_2 = M_0 \subseteq_a (M_1 \subseteq_b M_2)$

Note that $H_0(A_0) = \text{decatagorification of } \mathcal{A}_0$, so

the fact that \downarrow has assoc. mult. is a "tuning-down" of the fact that s.e.s.'s / 3-step filtrations are "assoc."

—X—

② So our aim now is to try and figure out what exactly $H_0(A_0)$ is. We start with a very easy example, and then ~~the~~ another example where A is NOT semisimple.

① $A_0 = \text{Fin Dim Vect}_F$ (any field F) $\Rightarrow \dots \Rightarrow H_0(A_0) = F[x]$

Then it turns out that

$(H_0(A_0), m) = \text{usual polynomial algebra}$

② $A_0 = \text{Fin Dim Reps of the quiver } Q = (\bullet \rightarrow \bullet) \text{ over a fixed ground field } F.$

We first note: quiver-reps = $(V \xrightarrow{H} W)$

Moreover, to classify these reps, it is enough to classify the indecomposables \rightarrow and here they are:

$F \xrightarrow{1} F, 0 \xrightarrow{0} F, F \xrightarrow{0} 0$ (of obtaining the Kull alg.)

③ NOTE This construction can be done for all quivers with fin. many indec. reps, and by a remarkable result, these are precisely the simply-laced Dynkin quivers (ADE). // And Kull alg is indep. of arrow-directions in this case.