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(1) (a) Recall, from last quarter, that de-groupoidification takes a groupoid to its zeroth homology.

$X \mapsto H_0(X) = \text{Free}(\text{components of } X)$

It also takes spans between groupoids to linear operators.

(b) We'll consider an example. Let $\mathcal{A}$ be any abelian category; so we consider $\mathcal{A}_0$.

Of course, ~~these~~ are closed under $\oplus = \Pi = \amalg$,
so ~~they~~ are never finite unless $\mathcal{C} = \{0\}$.

Essentially, all small abelian categories are $\cong$ R-Mod.

And the name comes from the prototype $\mathbb{Z}$-Mod = category of abelian groups.

But we'll try to not get "too infinite" eg. think of the following example:

$\mathcal{A} \equiv$ fin. dim. rep's of a finite quiver over a finite field

(c) The idea is to try to get linear operators: $V \times V \to V$
or equivalently, $V \otimes V \to V$

And this will come from a tri-span, ...

$$\begin{tikzcd} & Z & \\ & S \arrow[u] \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$ where all are groupoids

More precisely, we will de-groupoidify
$$\begin{tikzcd} & A_0 & \\ & ?? \arrow[u, "?"'] \arrow[dl] \arrow[dr] & \\ A_0 & & A_0 \end{tikzcd}$$

and get a "3-index tensor"
which can be thought of / re-interpreted
as a bilinear map: $V \otimes V \to V$
$(V = H_0(A_0)\ ?)$

d) In fact, $?? = SES(A_0)$
$\downarrow$
This is a category, and we have elts =
$[0 \to S \longrightarrow M \to Q \to 0 \text{ in } A]$
(sub) (mod) (quot)

Our tri-span now becomes
$$\begin{tikzcd} A_0 & & A_0 \\ & SES(A_0) \arrow[ul, "``S"'] \arrow[ur, "``Q"'] \arrow[d, "``M"'] & \\ & A_0 & \end{tikzcd}$$

Now, $\left[\begin{tikzcd} & \mathcal{C} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \mathcal{D} & & \mathcal{E} \end{tikzcd}\right] \longleftrightarrow \left[\begin{tikzcd} \mathcal{C} \arrow[d, "(a,b)"] \\ \mathcal{D} \times \mathcal{E} \end{tikzcd}\right]$

So $\left(\begin{tikzcd} & \mathcal{F} & \\ & \mathcal{C} \arrow[u] \arrow[dl] \arrow[dr] & \\ \mathcal{D} & & \mathcal{E} \end{tikzcd}\right) \cong \left(\begin{tikzcd} & \mathcal{C} \arrow[dl] \arrow[dr] & \\ \mathcal{D} \times \mathcal{E} & & \mathcal{F} \end{tikzcd}\right)$

[illegible]

and by our H_0 - Construction, this gives a linear map

$$H_0(\mathcal{D} \times \mathcal{E}) \longrightarrow H_0(\mathcal{F})$$

and $\downarrow \cong H_0(\mathcal{D}) \otimes H_0(\mathcal{E})$.

(e) So, UPSHOT: We seek, from $SES \rightrightarrows \mathcal{A}_0$,

a linear map:
$$\begin{array}{c} H_0(\mathcal{A}_0) \otimes H_0(\mathcal{A}_0) \\ \downarrow m \\ H_0(\mathcal{A}_0) \end{array}$$

AND, hope that this map m is "assoc", which would make $H_0(\mathcal{A}_0)$ into an assoc. alg.

In fact, this does turn out to be the case, and $H_0(\mathcal{A}_0)$ is called the Hall algebra of the abelian category $\mathcal{A}$.

(f) Here's a "vague" reason why we expect the mult. to be associative.

"Mult" three things is like a module of "length" 3, i.e. a module with a 3-stage filtration

$$\begin{array}{ccccccc} 0 & \subseteq M'' & \subseteq & M' & \subseteq M \\ \| & & & & \| \\ M_0 & \underset{a}{\subseteq} M_1 & \underset{b}{\subseteq} & M_2 & \underset{c}{\subseteq} M_3 \end{array}$$

But now assoc is $(ab)c = a(bc)$
i.e. $(M_0 \subseteq M_2) \subseteq M_3 = M_0 \subseteq (M_1 \subseteq M_3)$

Note ~~that~~ $H_0(\mathcal{A}_0)$ = decategorification of $\mathcal{A}_0$, so the fact that $\downarrow$ has assoc. mult., is a "toning-down" of the fact that s.e.s.'s / 3-step filtrations are "assoc.".

—X—

(2) So our aim now is to try and figure out what exactly $H_0(\mathcal{A}_0)$ is. We start with a very easy example, and then ~~B~~ another example where $\mathcal{A}$ is <u>NOT semisimple</u>.

(a) $\mathcal{A}_0$ = Fin Dim Vect$_F$ (any field F) $\Rightarrow \cdots \Rightarrow$ $H_0(\mathcal{A}_0) = F[X]$

Then it turns out that

$(H_0(\mathcal{A}_0), m)$ = usual polynomial <u>algebra</u>.

(b) $\mathcal{A}_0$ = Fin Dim Reps of the quiver $Q = (\circ \longrightarrow \circ)$ over a fixed ground field F.

We first note: quiver-reps = $(V \xrightarrow{H} W)$

<u>Moreover to classify these reps, it is enough to classify the <u>indecomposables</u></u> → and here they are:

$F \xrightarrow{1} F$, $0 \xrightarrow{0} F$, $F \xrightarrow{0} 0$

(of obtaining the Hall alg.)

(c) $\boxed{\text{NOTE}}$ This construction can be done for all quivers with fin. many indec. reps - and by a remarkable result, these are precisely the simply-laced Dynkin quivers (ADE). !! And Hall alg is indep of arrow-directions in this case.