4N/2808/THU 1

John Baez

1) (a) Recall from last time that we worked with the Harmonic Oscillator, and we then had the position and momentum operators

$$p, q : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \text{ given by}$$

$$q\psi(x) = x\psi(x)$$
$$p\psi(x) = \frac{1}{i}\psi'(x)$$

(b) Then $p, q$ are both self-adjoint, $[p,q] = \frac{1}{i}$

and now we define(d) the creation and annihilation operators

$$a^* = \frac{p+iq}{\sqrt{2}} \qquad a = \frac{p-iq}{\sqrt{2}}$$

And, $H = \frac{1}{2}(p^2+q^2) = a^*a + \frac{1}{2}$ is also self-adjoint

(c) Now, $\psi_0 \in L^2(\mathbb{R})$, given by $e^{-x^2/2}$, is an eigenfunction

$\left(H\psi_0 = \frac{1}{2}\psi_0 \Leftarrow\right) a\psi_0 = 0$ (easy to check)

Moreover, the eigenvalues of $H$ are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$ Though this seems a bit tedious mathematically, it is crucial in physics — the $\frac{1}{2}$ is the energy of the vacuum vector, arising out of the Uncertainty Principle.

(d) $N = H - \frac{1}{2}$ has eigenvalues $= \mathbb{N}_0 = \mathbb{Z}_{\geq 0}$; we'll work with this now. Then $[N, a^*] = a^*$, $[N, a] = -a$

and $N\psi = \lambda\psi \Rightarrow N(a^*\psi) = (\lambda+1)\, a^*\psi$
$N(a\psi) = (\lambda-1)\, a\psi$.

So if $\psi_n = (a^*)^n \psi_0$, then $N\psi_n = n\psi_n$.

This is why $N$ is called the <u>number operator</u>.

(e) Note that oscillators are present in many examples in physics — eg. light in a box. Then the energy states are discrete, and given by the number of <u>photons</u> in the box.
(So in the units where the energy of a photon $= 1$, we have vacuum energy $= \frac{1}{2}$.)

(f) ~~Note~~ $\boxed{\text{Theorem}}$ $\{\psi_n : n \in \mathbb{N}\}$ form an orthogonal <u>Hilbert space basis</u> of $L^2(\mathbb{R})$

[ie we have $L^2(\mathbb{R}) = \text{completion}(\text{span}(\{\psi_n : n\}))$]

(Essentially a Stone–Weierstrauss–type result.)

(g) $\boxed{\text{Note}}$: $a^*\psi_n = \psi_{n+1}$ ; $a\psi_n = n\,\psi_{n-1}$ $(\psi_{-1} = 0)$
and hence (easy to see)
$N\psi_n = a^*a\psi_n = n\psi_n$.

So we have an analogy : $(L^2(\mathbb{R}), \{\psi_n\}, a^*, a)$
and $(\mathbb{C}[z]\ \&\ F[z], \{z^n\}, \underbrace{(\cdot z)}_{m_z}, \frac{d}{dz})$

This means that we can define an operator

$$\alpha : \mathbb{C}[z] \xrightarrow{\ } L^2(\mathbb{R})$$

$$z^n \longmapsto \psi_n$$

Then $\begin{pmatrix} m_z & \longmapsto & a^* \\ d/dz & \longmapsto & a \end{pmatrix}$ and $\alpha$ is 1-1 with dense image

(h) $\boxed{\text{HW}}$ Find the unique inner product on $\mathbb{C}[z]$ so that $m_z$ and $d/dz$ are adjoint, and $\|z^1\| = 1$

With this inner product on $\mathbb{C}[z]$, $\mathbb{C}[z]$ becomes a <u>pre-Hilbert space</u>, and

$\alpha$ extends to $\tilde{\alpha} : \overline{\mathbb{C}[z]} \longrightarrow L^2(\mathbb{R})$, as a unitary operator.

(This is also called <u>Fock space</u>).

This is in fact what happens while studying physics. In quantum mechanics, one studies it via $L^2(\mathbb{R})$. In QFT, things are done in the Fock space. And from above, these approaches are equivalent!

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(2) Henceforth, we'll drop the analysis, replace $\mathbb{C}$ by any field (maybe with Char. 0) $k$, and work with $k[z]$ with operators $a^* = m_z$, $a = \frac{d}{dz}$, $N = a^* a$.

(a) We can easily generalize this to $n$ (types of) particles

$k[z_1, \ldots, z_n]$ with operators $a_i^* = m_{z_i}$, $a_i = \frac{\partial}{\partial z_i}$, $N_i = a_i^* a_i$,

$N$ = total number operator $= \sum_{i=1}^{n} N_i$

(b) But there are two other, relatively harder, manoeuvers (?) that we would like to do — groupoidification
— q-deformation.

First, let's groupoidify $(k[z], a, a^*)$. The more systematic ~~thing~~ thing is the reverse procedure, ~~take~~ namely

degroupoidification : ~~vector space~~

groupoids $\longrightarrow$ vector spaces
spans of groupoids $\longrightarrow$ linear operators

(c) The way to do this is $X \mapsto$ Free $(\underline{X}) =: H_0(X)$, the "zeroth homology".

where ~~if~~ if $X$ is a groupoid, there's a set $\underline{X}$ of isomorphism classes of objects, and we then form a vector space with $\underline{X}$ as basis.

As for a linear operator, here's how to get it: if we have a functor

$f: X \to Y$, then we get a linear operator

$$f_*: H_0(X) \to H_0(Y)$$
$$[x] \mapsto [f(x)]$$

and also sometimes, $f^! : H_0(Y) \mapsto H_0(X)$

$$f^! : [y] \longmapsto |\mathrm{Aut}(y)| \sum_{[x] \in f^{-1}(y)} \frac{[x]}{|\mathrm{Aut}(x)|}$$

(This is needed to make $\mathrm{id}^! = \mathrm{id}$ !)

Here, $f^{-1}(y)$ is the full sub groupoid of $X$ whose objects are $x \in X$ with $f(x) \cong y$

and $\mathrm{Aut}(x) = \{\text{all isomorphisms} : x \to x\}$

(d) There are two potential problems:

(i) The sum above may be infinite $\rightarrow$ maybe we can use Borel-Moore homology here, ~~to~~ allow us to write infinite sums.

(ii) $\mathrm{Aut}(X)$ may be infinite.

(e) So we'll avoid ~~these~~ ~~assum~~ problems in the examples that we study. We still have to worry about converting spans to linear operators.

$\boxed{\text{Example}}$ $(\mathrm{FinSet})_0$ has $H_0(\mathrm{FinSet}_0) = k[z]$.

and consider $\mathrm{FinSet}_0 \xrightarrow{+1} \mathrm{FinSet}_0$

Note: This really is a functor

$$\begin{array}{ccc} S & \xrightarrow{+1} & S \amalg \{*\} \\ \downarrow \alpha & & \downarrow \alpha+1 \\ T & \xrightarrow{+1} & T \amalg \{*\} \end{array}$$

What is $H_0(\text{FinSet}_0) \xrightarrow{(+1)_*} H_0(\text{FinSets})$ ?

<u>Ans:</u> $z^n \longmapsto z^{n+1}$, so its the <u>creation operator</u> $a^*$.

What's $H_0(\text{FinSet}_0) \xrightarrow{(+1)^!} H_0(\text{FinSet}_0)$ ?

Well, looking at the formula, if $[y] = z^n$, ie $y$ is any $n$-element set,

then $f^{-1}(y) = \{\text{all } (n-1)\text{-elements sets}\}$, so

$$(+1)^!(z^n) = (n!) \sum_{\text{one elements} = [x] = z^{n-1}} \frac{z^{n-1}}{(n-1)!} = n z^{n-1}$$

$\left[\text{because } |\text{Aut}(y)| = n!, \quad |\text{Aut}(x)| = (n-1)!\right]$

So this is the <u>annihilation operator</u> $a$.

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(3) $\boxed{\text{Homework}}$ $1 = \langle z^1, z^1 \rangle = \langle m_z(1), z \rangle = \langle 1, \frac{d}{dz} z \rangle = \langle 1, 1 \rangle = 1$

Now, $\forall n$, $\langle z^n, z^n \rangle = \langle m_z(z^{n-1}), z^n \rangle = \langle z^{n-1}, (z^n)' \rangle$
$= n \cdot \langle z^{n-1}, z^{n-1} \rangle$

Hence by induction, $\langle z^n, z^n \rangle = n! \quad \forall n$

and if $m > n$, $\langle z^m, z^n \rangle = \langle m_z(z^{m-1}), z^n \rangle = \langle z^{m-1}, n z^{n-1} \rangle$
$= n \langle z^{m-1}, z^{n-1} \rangle$
$= \cdots = n! \langle z^{m-n}, z^0 \rangle = n! \langle m_z(z^{m-n-1}), 1 \rangle$
$= n! \langle z^{m-n-1}, 0 \rangle = 0.$

$\boxed{\text{ANS}}$ $\boxed{\langle z^m, z^n \rangle = \delta_{m,n}\, n!}$, and extend by bilinearity (here, $m, n \in \mathbb{Z}_{\geq 0}$).