JAN/2008/1

Jim Dolan

1) (a) There is some parallelism between the notions of the Hall algebra and the annihilation operator - in that the definitions of both of them involving summing over a set of ~~facts~~ "possible outputs".

(b) Some Hall algebras (eg those coming from quivers over $\mathbb{F}_q$) are parts of quantum groups.

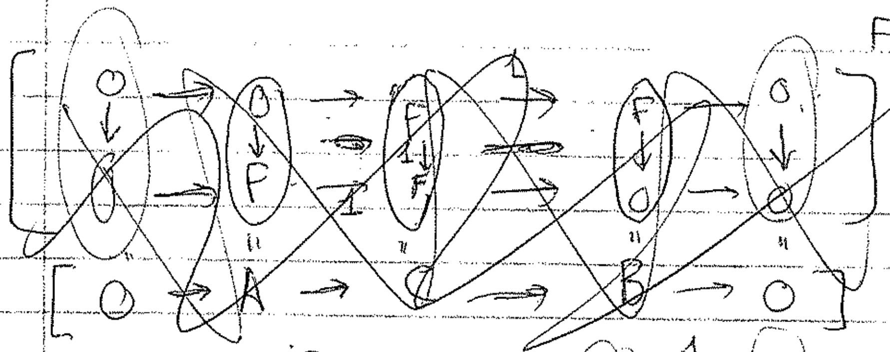
On the other hand, they are also the degroupoidification / decategorification of $Rep_{\mathbb{F}_q}(Q) = A$ (Q = quiver)

Combining these two leads us to think about the categorification of $U_q(n^+)$. Can we categorify $U_q(g)$-reps - which are, by extension, modules over the Hall algebra?

(2) (a) We go back to our specific example $[A_2 = \bullet \to \bullet]$

with 3 indecomposables

$F \to 0$ } irred
$0 \to F$
$F \xrightarrow{1} F$



$$\begin{bmatrix} 0 & \to & 0 & \to & F & \xrightarrow{1} & F & \to & 0 \\ \downarrow & & \downarrow & & \downarrow 1 & & \downarrow & & \downarrow \\ 0 & \to & F & \xrightarrow{1} & F & \to & 0 & \to & 0 \end{bmatrix}$$

$$\begin{matrix} \| & & \| & & \| & & \| & & \| \\ 0 & \to & A & \to & C & \to & B & \to & 0 \end{matrix}$$

Thus, we should have $\begin{bmatrix}0\\ \downarrow\\ F\end{bmatrix} \cdot \begin{bmatrix}F\\ \downarrow\\ 0\end{bmatrix} = \alpha_0 \begin{bmatrix}F\\ {\scriptstyle 0}\downarrow\\ F\end{bmatrix} + \cancel{\beta}\,\alpha_1 \begin{bmatrix}F\\ {\scriptstyle 1}\downarrow\\ F\end{bmatrix}$

in $H_0(\mathcal{A}_0)$, and we want ~~to~~ compute α_0, α_1.

(b) Recall our definition of the transfer map (here, we work over $\mathbb{Q}$)

G_1, G_2 = groupoid, F functor : $G_1 \longrightarrow G_2$

$\Rightarrow F^! : H_0(G_2) \longrightarrow H_0(G_1)$ is given by

$$F^!(Y) = \frac{\sum_{X \text{ a component of } F^{-1}(Y)} |X| \cdot X}{|Y|}$$

Y = component of G_2

($|X|$ = Groupoid Cardinality $= 1/|\mathrm{Aut}(X)|$)

Now, $G_1 = (\text{Category of S.E.S.'s in } \mathcal{A} = \mathrm{Rep}_{\mathbb{F}}(Q))_0$

$G_2 = (\mathcal{A} \times \mathcal{A})_0$

and our trispan / (bi-)span is

$$\begin{tikzcd} & (SES(\mathcal{A}))_0 = G_1 \arrow[dl, "{(S,Q)}"'] \arrow[dr, "M"] & \\ G_2 = (\mathcal{A} \times \mathcal{A})_0 & & \mathcal{A}_0 \end{tikzcd}$$

where S, Q, M take a ses $0 \to N' \to N \to N'' \to 0$ in $\mathcal{A}$ to N', N'', N respectively.

(c) So we now ~~or~~ work some things out w.r.t. the transfer map.

This gives $(S,Q)^!: H_0((\underline{\mathcal{A}}\times\underline{\mathcal{A}})_0) \to H_0(SES(\mathcal{A})_0)$

But $(\mathcal{A}\times\mathcal{A})_0 = \mathcal{A}_0\times\mathcal{A}_0$, and so

$H_0(\mathcal{A}_0\times\mathcal{A}_0) = H_0(\mathcal{A}_0)\otimes H_0(\mathcal{A}_0)$, whence we want to compute

$$(S,Q)^!: H_0(\mathcal{A}_0)\otimes H_0(\mathcal{A}_0)\to H_0(SES(\mathcal{A})_0)$$

Let's do an example — our example!

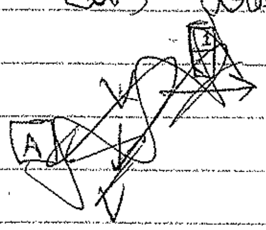
$$\begin{bmatrix}0\\ \downarrow\\ F\end{bmatrix}\otimes\begin{bmatrix}F\\ \downarrow\\ 0\end{bmatrix}\xrightarrow{(S,Q)^!} ?\left[\begin{array}{ccccc}0&\to&K&\xrightarrow{1}&K\\ \downarrow&&\downarrow 0&&\downarrow\\ K&\xrightarrow{1}&K&\to&0\end{array}\right]$$

$$+\ ?\left[\begin{array}{ccccc}0&\to&K&\xrightarrow{1}&K\\ \downarrow&&\downarrow 1&&\downarrow\\ K&\xrightarrow{1}&K&\to&0\end{array}\right]$$

(d) One of these coefficients ? is doable in full "generality" — the size of the Aut-gp for a split ses inside any abelian category.

Given a ses $V\to V\oplus W\to W$, what is Aut( )?

Let's draw it using isos $A: V\to V$, $F: W\to W$.



$$\begin{tikzcd} V \arrow[r, "{\begin{array}{c|c} & V \\ \hline V & 1 \\ W & 0 \end{array}}"] \arrow[d, "\boxed{A}"'] & V\oplus W \arrow[r, "{\begin{array}{c|cc} & V & W \\ \hline W & 0 & 1 \end{array}}"] \arrow[d, "{\begin{array}{c|cc} & V & W \\ \hline V & B & C \\ W & D & E \end{array}}"] & W \arrow[d, "\boxed{F}"] \\ V \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] & V\oplus W \arrow[r, "{\begin{array}{c|cc} W & 0 & 1 \\ \hline & V & W \end{array}}"'] & W \end{tikzcd}$$

And now we write our commuting squares:

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}\boxed{A} \quad\Rightarrow\quad \begin{array}{l} B = A \\ D = 0 \end{array}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} B & C \\ D & E \end{bmatrix} = \boxed{F}\begin{bmatrix} 0 & 1 \end{bmatrix} \quad\Rightarrow\quad \begin{array}{l} E = F \\ D = 0 \end{array}$$

So, $\mathrm{Aut}(V\oplus W) = |\mathrm{Aut}(V)|\cdot|\mathrm{Aut}(W)|\cdot|\mathrm{Hom}(W,V)|$

$\boxed{\text{Corollary}}$ In the Hall algebra in this case, in the expansion of $[V]\cdot[W]$, the coeff. of $[V\oplus W]$ is

$$\frac{|\mathrm{Aut}(V\otimes W)|}{|\mathrm{Aut}(V\oplus W)|} = \frac{|\mathrm{Aut}\cancel{(V)}|\cdot|\cancel{\mathrm{Aut}(W)}|}{|\cancel{\mathrm{Aut}(V)}|\cdot|\cancel{\mathrm{Aut}(W)}|\cdot|\mathrm{Hom}(W,V)|} = \boxed{\frac{1}{|\mathrm{Hom}(W,V)|}}$$

(e) Applying this to our example,

$$\begin{bmatrix}0\\\downarrow\\F\end{bmatrix}\otimes\begin{bmatrix}F\\\downarrow\\0\end{bmatrix} \longmapsto \frac{1}{\left|\mathrm{Hom}\left(\begin{matrix}F\\\downarrow\\0\end{matrix},\begin{matrix}0\\\downarrow\\F\end{matrix}\right)\right|}\cdot\left[\begin{array}{ccccc} 0 & \to & F & \xrightarrow{1} & F \\ \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow \\ F & \xrightarrow{1} & F & \to & 0\end{array}\right]$$

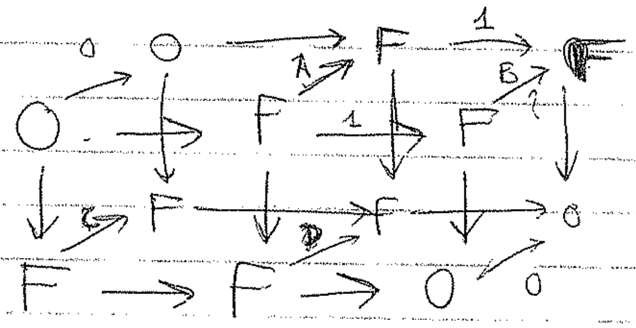
$$= \frac{1}{1} = 1$$

$$+ \ ? \cdot \left[\begin{array}{ccccc} 0 & \to & F & \xrightarrow{1} & F \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\ F & \xrightarrow{1} & F & \to & 0\end{array}\right]$$

For this, we need Aut's of

|| This will involve

a 3-dimensional picture:



But then it's easy to see (perhaps in a clearer diagram!) that

$$C = D = A = B$$

So, the coefficient is (same numerator)

$$\frac{\left|\mathrm{Aut}_F\begin{pmatrix}0\\\downarrow\\F\end{pmatrix}\right| \cdot \left|\mathrm{Aut}_F\begin{pmatrix}F\\\downarrow\\0\end{pmatrix}\right|}{\left|\mathrm{Aut}_F(F)\right|}$$

and check that this is

$$\frac{|F^\times|\,|F^\times|}{|F^\times|} = |F^\times|$$

So if $F = \mathbb{F}_q$, then

$$\underset{A}{\begin{bmatrix} 0 \\ \downarrow \\ F \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} F \\ \downarrow \\ 0 \end{bmatrix}} = 1 \cdot [A \oplus B] + (q-1) \cdot [C]$$

(3) (a) The sum of the coefficients is $1 + |F^\times| = |F|$.
Why so?

Well, there are two notions of identifying s.e.s's in $\mathcal{A}$. The one that we used was that of isomorphic s.e.s's, which meant only 2 such iso-classes

$[A \oplus B]$, $[C]$ of B by A.

On the other hand, the standard way is to consider the equivalence classes of s.e.s's — and these are classified by

$$|\operatorname{Ext}^1(W, V)_*|$$

which is, in this case, $|F|$!!

and then we should be summing over Ext^1, to compute in the Hall algebra. And this ~~why~~ ties in nicely with the q-elements above.