AN/2008

$\boxed{\text{John Baez}}$

1) (a) Given a span of groupoids

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

(or S), we get a linear operator

$$\tilde{S} : H_0(X) \to H_0(Y)$$

$$\tilde{S} = q_* p^!$$

(b) Given composable spans

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & & T \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ X & & Y & & Z \end{tikzcd}$$

we compose them via <u>weak pullback</u>

$$\begin{tikzcd} & TS \arrow[dl] \arrow[dr] & \\ S \arrow[dr, "q"'] & & T \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

$$TS := [(s \in S, t \in T), \alpha : q(s) \xrightarrow{\sim} p'(t)]$$

$\boxed{\text{Theorem}}$ $\widetilde{TS} = \tilde{T} \circ \tilde{S}$

(c) We also have an identity span

$$1_X = \begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X \end{tikzcd}$$

Then

$\boxed{\text{Theorem}}$ $\tilde{1}_X = 1_{H_0(X)}$

(d) <u>Adding spans?</u> Given spans

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \quad \& \quad \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd},$$

there's an obvious way to construct

$$\begin{tikzcd} & S \sqcup T \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

and this is denoted as $S+T$. So,

$$S+T = S \sqcup T,$$ with the same equation on the level of objects as well as morphisms!

Theorem ① $S+T \simeq T+S$ (as groupoid/spans)

② $\widetilde{S+T} = \widetilde{S} + \widetilde{T}$. $\left(= \widetilde{T} + \widetilde{S} = \widetilde{T+S} \right)$.

ⓔ <u>Remark</u>: In adding spans, we're taking "OR" (logic) — which is what happens in quantum mechanics when one applies ~~a~~ a linear operator to a state.

One gets <u>superposition of states</u> — and this is a good "analogy" since we <u>are</u> trying to groupoidify quantum mechanics.

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② ⓐ Back to the <u>Annihilation and Creation Operators</u>

We have: $H_0(\mathrm{FinSet}_0) \cong k[z]$ (k a ground field of char. 0)

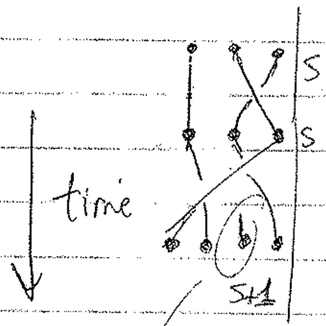
and certain spans (that groupoidify a, a^*):

$$A^* = \begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "id"'] \arrow[dr, "+1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd} \qquad A = \begin{tikzcd} & \mathrm{FinSet} \arrow[dl, "+1"'] \arrow[dr, "id"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

(b) Which we can draw as Feynman diagrams:

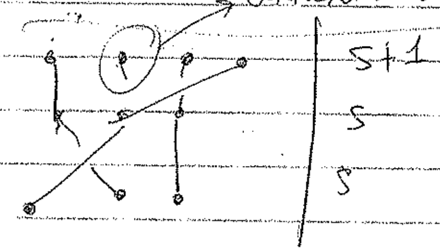
$$A^* = \begin{tikzcd} & \mathrm{FinSet}_0 \ni S \arrow[dl, "id"'] \arrow[dr, "+1"] & \\ \mathrm{FinSet}_0 \ni S & & \mathrm{FinSet}_0 \ni (S+1) \end{tikzcd} = $$

$\xrightarrow{\text{time}}$



[a fourth particle gets "created"]

Similarly, $A =$



(a) Now let's show: $AA^* \simeq A^*A + Id$.

(and thus, $\widetilde{AA^*} = aa^*$
$\qquad\qquad \| \qquad\qquad \|$
$\qquad \widetilde{A^*A + Id} = a^*a + Id$)

$\boxed{\text{Note}}$ We are using some more functorial properties, eg

$$S \simeq T \Rightarrow \tilde{S} = \tilde{T}, \text{ and:}$$

Moreover we have the notion of the adjoint span:

Given
$$\left[\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}\right] = S \text{ , the \underline{adjoint span} is defined}$$

to be
$$\left[\begin{tikzcd} & S^*(=S) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}\right] = S^*$$

Then $\boxed{\text{Thm}}$ (1) All $H_0(X)$'s have inner products on them, whence linear operators have <u>adjoints</u>!

(2) Thus, we have $H_0(X) \xrightarrow{\tilde{S}} H_0(Y)$

$$\Downarrow$$

$$H_0(Y) \xrightarrow{(\tilde{S})^*} H_0(X)$$

and we also have $H_0(Y) \xrightarrow{\widetilde{(S^*)}} H_0(X)$.

<u>Then</u>, $\tilde{S}^* = \widetilde{S^*}$

(3) $(S^*)^* = S$, $(\tilde{S}^*)^* = \tilde{S}$.

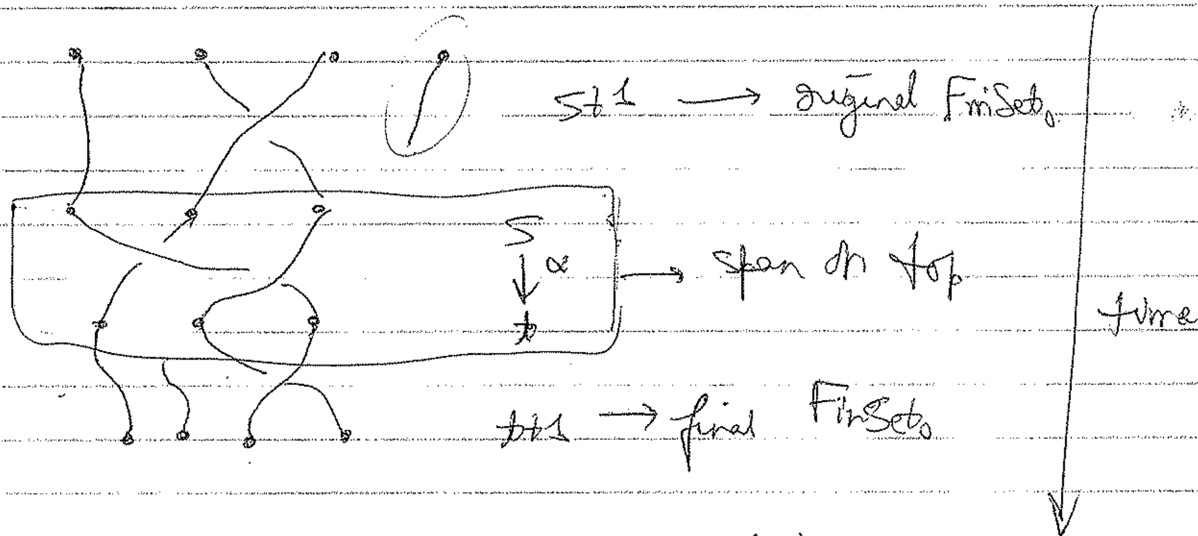
(c) We start, now, with A^*A (the "easy composite"):

$$\begin{tikzcd} & & \mathrm{FinSet}_0 \arrow[dl, "id"'] \arrow[dr, "id"] & & \\ & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "id"] & & \mathrm{FinSet}_0 \arrow[dl, "id"'] \arrow[dr, "+1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

The groupoid on top is the weak pullback:

$$[s,t \in \mathrm{FinSet}_0,\ \alpha : s \xrightarrow{\sim} t] \quad \text{—————} \quad (*)$$
$$\Vert ?$$
$$\mathrm{FinSet}_0.$$

Let's now draw the Feynman diagram, but not using our simplification in (*), but using 'pairs of (isom.) finite sets'.



This is how the composite span's "elements" look.

$$\begin{tikzcd}[column sep=small]
 & & {[s,t \in \mathrm{FinSet}_0,\ \alpha : s+1 \xrightarrow{\sim} t+1].} \arrow[dl, "(s,t,\alpha) \mapsto s"'] \arrow[dr, "(s,t,\alpha) \mapsto t"] & & \\
 & \mathrm{FinSet}_0 \arrow[dl, "id"'] \arrow[dr, "+1"] & & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "id"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

(d) To show our desired commutation relation on the groupoid level, we ~~start to~~ first want to write AA^* as a sum of two spans:

Before we do this on the span level, let's do it at the groupoid level on top only:

$$[s,t \in \mathrm{FinSet}_0,\ \alpha: s+1 \xrightarrow{\sim} t+1]$$

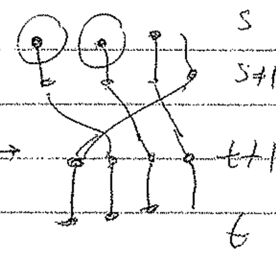
$$\simeq \ [s,t \in \mathrm{FinSet}_0 : \alpha: s \xrightarrow{\sim} t]$$

$$+ \ [s,t \in \mathrm{FinSet}_0,\ \alpha: s+1 \xrightarrow{\sim} t+1 \text{ s.t. } \alpha(1) \neq 1]$$

where $\swarrow$ $\simeq [s', t' \in \mathrm{FinSet}_0 : \alpha: s' \xrightarrow{\sim} t']$

$\simeq \mathrm{FinSet}_0$

$$\left[\text{Here, } \begin{array}{l} s' = s - \alpha^{-1}(1) \\ t' = t - \alpha(1) \end{array} : \right] \longrightarrow$$



(e) Now on the span level:

AA^* is

$$\begin{tikzcd}
& {[s,t \in \mathrm{FinSet}_0,\ \alpha: s+1 \xrightarrow{\sim} t+1]} \arrow[dl, "{(s,t,\alpha) \mapsto s}"'] \arrow[dr, "{(s,t,\alpha) \mapsto t}"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

This span is equivalent to the sum:

$$
\begin{tikzcd}
& \left[s, t \in \mathrm{FinSet}_0,\ \alpha : s \xrightarrow{\sim} t \right] \arrow[dl, "(s,t,\alpha) \mapsto s"'] \arrow[dr, "(s,t,\alpha) \mapsto t"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}
$$

$$
\left(\text{and this is } \simeq \quad
\begin{tikzcd}
& \mathrm{FinSet}_0 \arrow[dl, "id"'] \arrow[dr, "id"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}
= id \right)
$$

$$+$$

$$
\begin{tikzcd}
& \left[s', t' \in \mathrm{FinSet}_0,\ \alpha : s' \xrightarrow{\sim} t' \right] \arrow[dl, "(s',t',\alpha) \mapsto s'+1"'] \arrow[dr, "(s',t',\alpha) \mapsto t'+1"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}
$$

(which <u>is</u> the span A^*A)

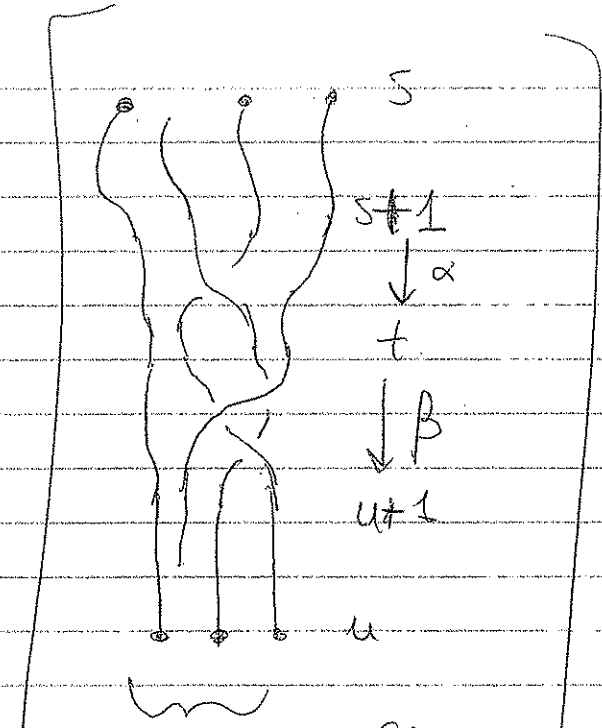
Hence the sum AA^* of these spans is equivalent to

$$1_{\mathrm{FinSet}_0} + A^*A$$

(Proved)

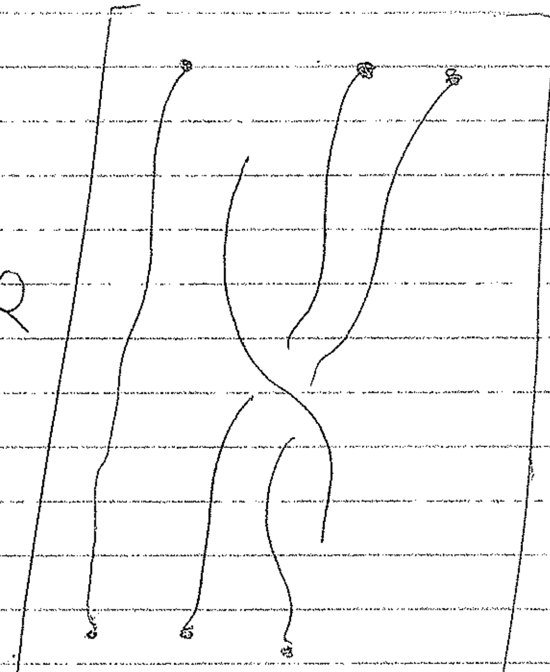
(f) OR VIA <u>Feynman diagrams</u>:

AA^* comes in two cases:



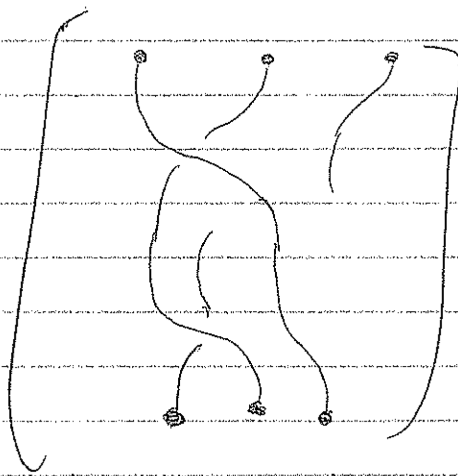
This is A^*A (?).

OR

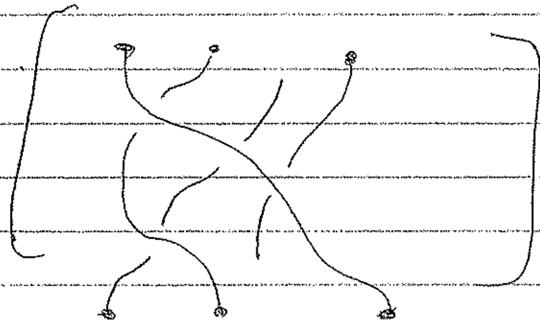


This is 1_{FinSet_0}

Well, A^*A really is



But one can just make ~~the~~ arrows longer:



→ and this is what we essentially drew at the top of this page.