We'll continue our journey towards representations of the Hall algebra — we may call these Hall modules. (We secretly remember that quantum groups are also on this picture.)

But first, let's try to show that the multiplication in the Hall algebra is associative:

\[ V \otimes V \rightarrow V \]

\[ V \otimes V \otimes V \rightarrow V \otimes V \]

\[ V \otimes V \rightarrow V \]

Comes from the sequence

\[ \text{SES}(A)_0 \]

or from the span

\[ \text{SES}(A)_0 \]

\[ (S, \varnothing, M) \]

\[ A_0 \times A_0 \]

\[ A_0 \]

(Here, \( A \) is any abelian category where the Hom- and Ext- sets are all finite — and we can consider \( A = \text{Rep}_k(A) \) if we wish. Moreover, define \( V = \text{Hol}(A_0) \).)

We can "groupify" the commuting square into a "square of squares".
If we can show that this "commutes," then by John B's last time functional properties of

\[ S \rightarrow [S : H_0 \rightarrow H_1] \]

directly pass that the square for the V's commutes

\[ \forall \sigma : H \rightarrow (H \times H) = H \times (H \times H) \]

Well, let's define for \( n \in \mathbb{N} \) the set \( n-SF(M) \)

\[ = n \text{-stage flags of any object } M \text{ with } A \text{; } nSF(A) \text{ class of all } \text{nSFIM} \]

Then \( 2-SF(A) = SES(A) \).

Now we want to say that the two composite spans

\[ A_0 \times A_0 \times A_0 \rightarrow A_0 \text{ actually agree/are \( \Rightarrow \)!} \]
Here's a pictorial key: 
- □ = our groupside
- △ = spans between them
- ○ = historically weak pullbacks used to compose spans.

Then we have to draw

For better notation, let \[ n' = (n - \text{SEP}(A)) \]

Then \( \mathcal{X}_0 = \square \) and \( \text{SES}(A)_0 = \triangle \)

So let's now compute #s 1 & 2.

(And note: here, we want to claim that first of all, both □ = [3]!)}
Here, \( \alpha \colon (B \leq C, D) \rightarrow (C, D) \)
\( \beta \colon (E, E) \rightarrow (E, E/E) \)

Then one can show that \( \cap = \sum B \leq C \rightarrow E \leq F \)
(So \( D = E/E = \) top part).
And thus, we have

\[ \gamma(B \leq C \leq F) = (B \leq C, F/E) \]
\[ \delta(B \leq C \leq F) = (C \leq F) \]

Here, \( \alpha' \colon (B \leq C) \rightarrow B, C/E \)
\( \beta' \colon (D, E, F) \rightarrow (D, E) \)

\( \gamma'(B \leq C \leq F) = (B \leq C) \)
\( \delta'(B \leq C \leq F) = (B, E \leq F/E) \)
And then one shows that the rest if it holds too!

The \( \mathbb{C} \cong \mathbb{D} \) is \( \mathbb{D} \) so we have just proved the groupoidified version of associativity!

(For any such abelian category \( \mathbb{A} \))

And now, by the functionality of degroupification to the multi-m.

\( \text{Ho}(\mathbb{A}) \text{ is also associative.} \)

\[
\times
\]

\section{2}

Now for \underline{all} Modules. Let's consider right-modules.

\textbf{Example} \quad \( A = \text{Rep}_{F_q}(\mathbb{A}) \)

\section{3}

Define, for any fun dim \( W = F_q \)-v.s.,

\[
B(W) := \begin{array}{c}
3 \text{-stage flags} \quad V_1 \hookrightarrow V_2 \hookrightarrow W \\
(S_0 \quad [V_1 \hookrightarrow V_2 \hookrightarrow W] \quad [V_1 \hookrightarrow V_2])
\end{array}
\]

is the forgetful functor \( B(W) \rightarrow A \).

(\text{TD calls these the \textit{legs} of \( A \) monically over \( W \)})

\section{4}

The action map then again comes from a \textit{tri-span} - and

proving that \( \text{Ho}(B(W)) \) is an \( \text{Ho}(\mathbb{A}) \)-module \( W \)

is similar to the above proof if associativity of \( \text{Ho}(\mathbb{A}) \).

\[
\]
And what is \( \dim H_0(B(W)) \)? It depends on \( \dim W \)

\[ \dim W = 1 \implies \text{upto mono,} \quad 0 \leq 0 \leq 1 \]
\[ 0 \leq 1 \leq 1 \implies \text{are the} \]
\[ 1 \leq 1 \implies \text{dim vectors} \]

In general, \( \# (m_1 \leq n_2 \leq \dim W) = ? \)

\[
\frac{(\dim W + 1)}{2}
\]

Finally, we draw the tri-span in question: Right module

\[ \mathcal{H}_0(B(W)) \otimes \mathcal{H}_0(U_0) \to \mathcal{H}_0(B(W)) \]

Where \( \varphi \) is all set of the kind

\[ B(W) \cong \begin{cases} S_1 \to S_2 \to W \\ \varphi \end{cases} \]

\[ B(W) \cong \begin{cases} M_1 \to M_2 \to W \\ \varphi \end{cases} \]

Only in \( U \mid \exists \phi : A_1 \to A_2 \to \phi \)