Jim Dolan (Hall Algebras – Cont^d)

ⓐ We'll continue our journey towards representations of the Hall algebras – we may call these Hall modules. (We secretly remember that quantum groups are also in this picture.)

ⓑ But first, let's try to show that the multiplication in the Hall algebra is associative:

$$V \otimes V \xrightarrow{m} V$$

$$\begin{tikzcd} V\otimes V\otimes V \arrow[r, "m\otimes 1_V"] \arrow[d, "1_V\otimes m"'] & V\otimes V \arrow[d, "m"] \\ V\otimes V \arrow[r, "m"] & V \end{tikzcd}$$

Comes from the trispan

$$\begin{tikzcd} \mathcal{A}_0 & & \mathcal{A}_0 \\ & SES(\mathcal{A})_0 \arrow[ul, "\text{``}S\text{''}"] \arrow[ur, "\text{``}Q\text{''}"'] \arrow[d, "\text{``}M\text{''}"] & \\ & \mathcal{A}_0 & \end{tikzcd}$$

or then from the span

$$\begin{tikzcd} & SES(\mathcal{A})_0 \arrow[dl, "{(S,Q)}"'] \arrow[dr, "M"] & \\ \mathcal{A}_0\times\mathcal{A}_0 & & \mathcal{A}_0 \end{tikzcd}$$

(Here, $\mathcal{A}$ is any abelian category where the Hom- & Ext^1-sets are all finite – and we can consider $\mathcal{A} = Rep_{\mathbb{F}_q}(A_2)$ if we wish. Moreover, define $V = H_0(\mathcal{A}_0)$.)

ⓒ We can "groupoidify" the commuting square for V's, into a "square" of spans:

~~$SES(\mathcal{A})_0 \times \mathcal{A}$~~

~~$\mathcal{A}\times\mathcal{A}\times\mathcal{A}$~~

$$
\begin{tikzcd}
 & SES(\mathcal{A})_0 \times \mathcal{A}_0 \arrow[dl] \arrow[dr] & \\
\mathcal{A}_0 \times \mathcal{A}_0 \times \mathcal{A}_0 & & \mathcal{A}_0 \times \mathcal{A}_0 \\
\mathcal{A}_0 \times SES(\mathcal{A}_0) \arrow[u] \arrow[d] & & SES(\mathcal{A}_0)_0 \arrow[u] \arrow[d] \\
\mathcal{A}_0 \times \mathcal{A}_0 & & \mathcal{A}_0 \\
 & SES(\mathcal{A})_0 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

(d) If we can show that this "commutes", then by John B.'s class last time, functorial properties of $S \longmapsto [S : H_0 \to H_0]$ directly prove that the square for the V's commutes! (Use: $H_0(\mathcal{A}\times\mathcal{B}) = H_0(\mathcal{A})\otimes H_0(\mathcal{B})$)

(e) Well, let's define for $n \in \mathbb{N}$, the set $n\text{-}SF(M)$

= n-stage flags of any object M in $\mathcal{A}$; $nSF(\mathcal{A})$ = class of all $nSF(M)$'s

Then $2\text{-}SF(\mathcal{A}) = SES(\mathcal{A})$.

Now, we want to say that the two composite spans

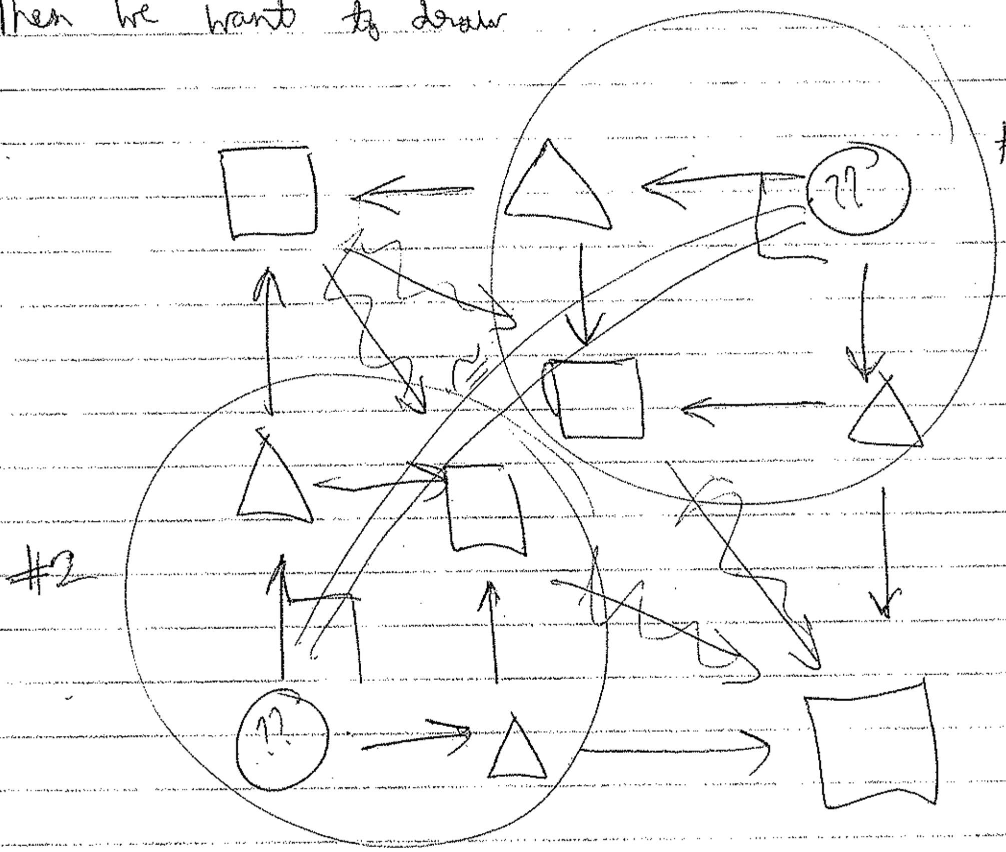
: $\mathcal{A}_0 \times \mathcal{A}_0 \times \mathcal{A}_0 \longrightarrow \mathcal{A}_0$ actually agree/are $\cong$!

Here's a pictorial key: □ = our groupoids

△ = spans between them

○ = homotopy/weak pullbacks used to compose spans.

Then we want to draw



(f) $\boxed{\text{OR}}$ For better notation, let $\boxed{\boxed{n} := (n\text{-SF})(\mathcal{A})_0}$

Then $\mathcal{A}_0 = \boxed{1}$, & $SES(\mathcal{A})_0 = \boxed{2}$!

So let's now compute #'s 1 & 2.

(and note: here, we want to claim that first of all, both ?? $= \boxed{3}$!)

$\boxed{\#1}$

$$\begin{tikzcd}
 & ?? \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
\boxed{2} \times \boxed{1} \arrow[dr, "\alpha"'] & & \boxed{2} \arrow[dl, "\beta"] \\
 & \boxed{1} \times \boxed{1} &
\end{tikzcd}$$

Here, $\alpha : (B \subseteq C, D) \longmapsto (C, D)$
$\beta : (E \subseteq F) \longmapsto (E, F/E)$

Then one can show that $?? = \{ B \subseteq C \xrightarrow{\sim} E \subseteq F \}$
$= \boxed{3}$

(So $D = F/E$ = top factor)

and the maps are

$\gamma(B \subseteq C \subseteq F) = (B \subseteq C, F/C)$
$\delta(B \subseteq C \subseteq F) = (C \subseteq F)$

$\boxed{\#2}$

$$\begin{tikzcd}
 & ?? \arrow[dl, "\gamma'"'] \arrow[dr, "\delta'"] & \\
\boxed{2} \arrow[dr, "\alpha'"'] & & \boxed{1} \times \boxed{2} \arrow[dl, "\beta'"] \\
 & \boxed{1} \times \boxed{1} &
\end{tikzcd}$$

Here, $\alpha' : (B \subseteq C) = (B, C/B)$
$\beta' : (D, E \subseteq F) = (D, E)$

So $?? = \{ B \subseteq C \xrightarrow{\sim} E \subseteq F \}$
$= \boxed{3}$

(so $D = B$ = bottom part)

and the maps are

$\gamma'(B \subseteq C \subseteq F) = (B \subseteq C)$
$\delta'(B \subseteq C \subseteq F) = (B, C/B \subseteq F/B)$

The (??) ≅ (??) and then one shows that the rest of it holds too! is $\underline{\underline{Id}}$.) So we have just proved the groupoidified version of associativity! (for any such abelian category $\mathcal{A}$)

And now, ~~if~~ by the functoriality of degroupoidification, ~~so~~ the mult. in $H_0(\mathcal{A}_0)$ is also associative.

—X—

(2) Now for Hall Modules. Let's consider right-modules.

$\boxed{\text{Example}}$: $\mathcal{A} = \cancel{\text{Reps}}\ \mathrm{Rep}_{\mathbb{F}_q}(A_2)$

(a) Define, for any fin dim $W = \mathbb{F}_q$-v.s.,

$$\mathcal{B}(W) := [\text{3-stage flags } V_1 \hookrightarrow V_2 \hookrightarrow W]$$

$$(\text{So } [V_1 \hookrightarrow V_2 \hookrightarrow W] \longrightarrow [V_1 \hookrightarrow V_2])$$

is the forgetful functor : $\mathcal{B}(W) \to \mathcal{A}$.

(JD calls these the "reps of A_2 monically over W")

(b) The action map then again comes from a tri-span — and proving that $H_0(\mathcal{B}(W)_0)$ is an $H_0(\mathcal{A}_0)$-module $\forall W$ is similar to the above proof of associativity of $H_0(\mathcal{A}_0)$.

(c) And what is $\dim H_0(\mathcal{B}(W)_0)$? It depends on $\dim W$,

e.g. $\dim W = 1 \Rightarrow$ (upto isom's), $0 \le 0 \le 1$, $0 \le 1 \le 1$, $1 \le 1 \le 1$ $\Big\}$ are the dim-vectors

In general $\#(n_1 \le n_2 \le \dim W) = ?$

$$\boxed{\text{Ans}} \quad \binom{(\dim W)+1}{2}$$

(d) Finally, we draw the tri-span in question: Right module,

so $H_0(\mathcal{B}(W)_0) \otimes H_0(\mathcal{A}_0) \longrightarrow H_0(\mathcal{B}(W)_0)$

So

$$\begin{tikzcd}
\mathcal{B}(W)_0 & & \mathcal{A}_0 \\
& SES(\mathcal{B}(W), \mathcal{A})_0 \arrow[ul, "\text{``}S\text{''}"'] \arrow[ur, "\text{``}Q\text{''}"] \arrow[d, "\text{``}M\text{''}"] & \\
& \mathcal{B}(W)_0 &
\end{tikzcd}$$

where is all ses of the kind

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathcal{B}(W) \ni & S_1 \arrow[r, hook] \arrow[d] & S_2 \arrow[r, hook] \arrow[d] & W \arrow[d, equal] \\
\mathcal{B}(W) \ni & M_1 \arrow[r, hook] \arrow[d] & M_2 \arrow[r, hook] \arrow[d] & W \arrow[d] \\
\text{only in } \mathcal{A}! \ni & Q_1 \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 & 0 & 0
\end{tikzcd}$$