3/28/08

~~[illegible]~~ John Baez – The Harmonic Oscillator (n deg. of freedom)

1) (a) We'll groupoidify representations of ~~the~~ GL_n, which comes into play once we consider the harmonic oscillator with n-degrees of freedom.

(b) On Thursdays Jim is talking about the Hall Algebra arising from $\mathrm{Rep}_{\mathbb{F}_q}(A_2)$, which generalizes to all A_n as well.

(c) As we'll see/say later, q-deforming (b) $\Rightarrow$ (c)
(this has to do with the fact that the Hall Alg. is a "part" $= U_q(\mathfrak{n}^+)$ of the q-deformed UEA $U(\mathfrak{gl}_n)$).

—X—

(2) So let us first (easily) generalize the theory of the Harmonic Oscillator to n-variables. So, we have

(a) $H_0(\mathrm{FinSet}_0^n) \cong H_0(\mathrm{FinSet}_0)^{\otimes n} \cong k[z]^{\otimes n}$

$$\cong k[z_1, \ldots, z_n]$$

where the iso class of $(k_1, \ldots, k_n) \in \mathrm{FinSet}_0^n$ corresponds to $z_1^{k_1} \cdots z_n^{k_n} \in k[z_1, \ldots, z_n]$.

We have n creation and annihilation operators, whose groupoidified versions are spans:

(b) Given i, denote $\underset{\sim}{e_i} = (\emptyset, \emptyset, \ldots, \{*\} = 1, \emptyset, \ldots, \emptyset) \in \mathrm{FinSet}_0^n$

i^{th} place

$(+1)_i$

Then the creation operator corresponds to $[\underset{\sim}{k} \mapsto \underset{\sim}{k} \amalg \underset{\sim}{e_i}]$

i.e. $(k_1, \ldots, k_n) \mapsto (k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n)$

So, we have:

A_i^* $(1 \leq i \leq n)$ is

$$\begin{tikzcd}
& \mathrm{FinSet}_0^n \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "(\underset{\sim}{k} \mapsto \underset{\sim}{k} \amalg \underset{\sim}{e_i})"] & \\
\mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n
\end{tikzcd}$$

This "creates a particle of type i".

A_i is

$$\begin{tikzcd}
& \mathrm{FinSet}_0^n \arrow[dl, "(+1)_i = (\underset{\sim}{k} \mapsto \underset{\sim}{k} \amalg \underset{\sim}{e_i})"'] \arrow[dr, "\mathrm{Id}"] & \\
\mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n
\end{tikzcd}$$

This annihilates a particle of type i.

(c) These give operators $\widetilde{A}_i = a_i$, $\widetilde{A}_i^* = a_i^*$ from the

Fock space $= k[z_1, \ldots, z_n]$

to itself, and explicitly, $\boxed{\begin{aligned} a_i &= \partial / \partial z_i \\ a_i^* &= m_{z_i} \end{aligned}}$

These satisfy: $a_i a_j = a_j a_i$

$a_i^* a_j^* = a_j^* a_i^*$

$a_i a_j^* = a_j^* a_i + \delta_{ij} \cdot 1$

These are called the "$\underline{\text{CCR}}$" = "$\underline{\text{canonical commutation relns}}$"

So, we hope that it also holds on the groupoid level

$$A_i A_j = A_j A_i$$
$$A_i^* A_j^* = A_j^* A_i^* \quad \text{(this follows from the previous line)}$$

$$A_i A_j^* = A_j^* A_i + \delta_{ij} 1 \quad \text{(this is already done for } i=j\text{)}$$

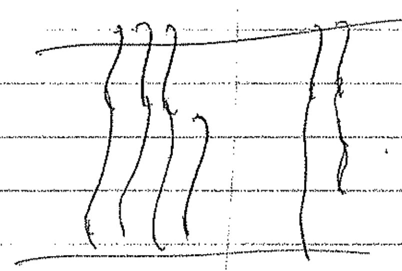
(d) Pf of: Let's first do the last line only, and for $i \neq j$

The span is computed to be

(i) $A_i A_j^*$

$$\begin{tikzcd}[column sep=small]
 & & & \left[(k_1,\dots,k_n),(l_1,\dots,l_n), \begin{array}{l}\alpha_j : k_j+1 \xrightarrow{\sim} l_j \\ \alpha_i : k_i \xrightarrow{\sim} l_i+1 \\ \alpha_s : k_s \xrightarrow{\sim} l_s \text{ o/w}\end{array}\right] \arrow[dll] \arrow[drr] & & & \\
 & \mathrm{FinSet}_0^n \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "(+1)_j"] & & & & \mathrm{FinSet}_0^n \arrow[dlll, "(+1)_i"'] \arrow[dr, "\mathrm{Id}"] & \\
\mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n & & & & \mathrm{FinSet}_0^n
\end{tikzcd}$$

In Feynman diagrams, we have Red Blue



(ii) $A_j^* A_i$ =

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{FinSet}_0^n \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & & \\
 & \mathrm{FinSet}_0^n \arrow[dl, "(+1)_i"'] \arrow[dr, "\mathrm{Id}"] & & \mathrm{FinSet}_0^n \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "(+1)_j"] & \\
\mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n
\end{tikzcd}$$

Why are these the same? It's because the ~~…~~ fact: composite span is $\cong$

$$\begin{tikzcd}[column sep=small] & \left[\,(\min(k_s, l_s))_{1\le s\le n}\,\right] \arrow[dl, "(+1)_i"'] \arrow[dr, "(+1)_j"] & \\ {} & & {} \end{tikzcd}$$

$$= \begin{tikzcd}[column sep=small] & \mathrm{FinSet}_0^n \arrow[dl, "(+1)_i"'] \arrow[dr, "(+1)_j"] & \\ \mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n \end{tikzcd}$$

and this makes it really easy to see that the desired CCR holds.

(e) We'll make more remarks on this after the notes for this class.

---

(3) So, we've groupoidified all these things:

(a) The assoc alg W_n generated by the a_i, a_i^* $(i=1,n)$

this is called the ~~…~~ n^{th} Weyl algebra.

This has a god-given rep. on $k[z_1, \ldots z_n]$ with

$$a_i \mapsto \partial/\partial z_i \quad , \quad a_i^* \mapsto m_{z_i}$$

A typical element in this rep. is like $z_1^3 \dfrac{\partial}{\partial z_2} + z_2 z_3 \dfrac{\partial^2}{\partial z_2 \partial z_1}$

So, $W_n = \{$polynomial coeff. diff. operators$\}$

This becomes a Lie algebra w/. $[a,b] = ab - ba$ and then it has lots of Lie subalgebras, including:

(b) {Polynomial coeff vector fields} = Diff^1

$$= \bigoplus k[z_1, \dots, z_n] \cdot \bigoplus k\left(\partial/\partial z_i\right)$$

(c) {Homogeneous linear coeff vector fields} $\cong gl(n)$, ie.

$$\text{eg } \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix} \mapsto 4z_1 \partial/\partial z_1 + 2z_1 \partial/\partial z_2 + 3z_2 \partial/\partial z_2$$

(d) Note that $[f \cdot \partial^{\underset{\sim}{n}}, g \cdot \partial^{\underset{\sim}{m}}] = f \cdot \partial^{\underset{\sim}{n}}(g)\partial^{\underset{\sim}{m}} - g \cdot \partial^{\underset{\sim}{m}}(f)\partial^{\underset{\sim}{n}}$

$\forall f, g, \underset{\sim}{n}, \underset{\sim}{m}$. So this explains why (c) works, ~~this~~ and can be generalized.

(e) The Heisenberg Lie Algebra $h_n =$ ~~set~~ Span of $(a_i, a_i^*, 1)$

This is a $(2n+1)$-dim Lie alg.

(f) {Homogeneous quadratic expressions in a_i, a_i^*, plus const}

~~Is this~~ ie spanned by $a_i^* a_j$ $\forall$
$a_i a_j$ $i < j$
$a_i^* a_j^*$ $i < j$
1

(g) $\boxed{\text{Qn}}$ Find commutation relations.
Is there a codim 1 Lie alg $\cong sp(2n)$?

ANS: One can check that (i) $[a_i^* a_j, a_k^* a_l] = \delta_{jk} a_i^* a_l - \delta_{il} a_k^* a_j$

(ii) $[a_i a_j, a_k^* a_l] = \delta_{jk} a_i a_l + \delta_{ik} a_j a_l$

(iii) $[a_i^* a_j^*, a_k^* a_l] = -\delta_{il} a_j^* a_k^* - \delta_{jl} a_i^* a_k^*$

And finally, $[a_i a_j, a_k^* a_l^*] = \delta_{jk} a_l^* a_i + \delta_{ik} a_l^* a_j + \delta_{jl} a_k^* a_i + \delta_{il} a_k^* a_j + \delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}$

This is where scalars may occur!

(h) How to amend? Well, change the basis/vectors from

$a_i^* a_j$ to $B_{ij} = a_i^* a_j + \frac{1}{2}\delta_{ij}\ \forall i,j$ (so clear $k \neq l$)

Then (i) above becomes $[B_{ij}, B_{kl}] = [a_i^* a_j, a_k^* a_l]$ ($\because 1$ is central)

$= \delta_{jk} a_i^* a_l + \frac{1}{2}\delta_{jk}\delta_{il}$

$- \delta_{il} a_k^* a_j - \frac{1}{2}\delta_{jk}\delta_{il}$

i.e. (i)': $[B_{ij}, B_{kl}] = B_{il} - B_{kj}$

(ii) Stays the same, and similar to (iii) changes to:

(iii)': $[a_i a_j, a_k^* a_l^*] = \delta_{jk} B_{li} + \delta_{ik} B_{lj} + \delta_{jl} B_{ki} + \delta_{il} B_{kj}$

and these span the symplectic Lie algebra (Proved)

④ (Continuing ②ⓔ above)

ⓐ Firstly, an ERRATUM: the weak pullback over ~~B~~ X, Y, ~~Z~~ is the terminal object, not the initial one, in the category of objects over X, Y, Z.

ⓑ Now, here is a slightly general criterion that can be used to prove ~~the~~ some of the CCR's (i.e. all the CCR's with commuting spans).

$\boxed{\text{Defn}}$ A functor is faithful and reflects isomorphisms ~~of~~ ($F: \mathcal{C} \to \mathcal{D}$)

if ⓐ $F(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective for all objects X, Y in $\mathcal{C}$

and ⓑ whenever $F(X) \cong F(Y)$ in $\mathcal{D}$, we must have $X \cong Y$ in $\mathcal{C}$

ⓒ $\boxed{\text{Lemma}}$ Say $\mathcal{C}$ is a category with functors $F, G: \mathcal{C} \to \mathcal{C}$ ~~that commute~~ ~~and are both isomorphism~~ (i) are faithful, (ii) reflect isomorphisms, and (iii) commute.

Also say that whenever $F(x) \xrightarrow{\sim} G(y)$ in $\mathcal{C}$, we must have $x \cong G(z)$, $y \cong F(z')$ for some z, z' in $\mathcal{C}$.

Then the weak pullback of $\mathcal{C} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{C}$

is

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "G"'] \arrow[dr, "F"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

In particular, the two spans below commute :

(and this implies most of the CCR's above).

$$\begin{tikzcd}[column sep=small] & \mathcal{C} \arrow[dl, "1"'] \arrow[dr, "F"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd} \quad , \quad \begin{tikzcd}[column sep=small] & \mathcal{C} \arrow[dl, "G"'] \arrow[dr, "1"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

(d) $\boxed{\text{Proof}}$ We first show that the weak pullback is, indeed, $\mathcal{C}$.

The weak pullback is $\{(x, y) \in Ob(\mathcal{C}) \times Ob\,\mathcal{C} \,;\ \alpha : F(x) \xrightarrow{\sim} G(y)\}$

By the given cond'n, this gives $(z, z') \in \mathcal{C}$ with

$$G(z) \cong x, \quad F(z') \cong y.$$

But then $F(x) \cong F(G(z)) = FG(z)$

$$\alpha \Downarrow \wr$$

$$G(y) \cong G(F(z')) = GF(z') = FG(z')$$

we can weaken this!

i.e. $FG(z) \cong FG(z')$. Since F, G both reflect isom's,

$z \cong z'$ in $\mathcal{C}$, i.e. $(x, y) \cong (G, F)(z)$

$\boxed{\text{CHECK}}$ This can be used to construct an equivalence between $\mathcal{C}$ and the weak pullback. (By universality, the functor in the other direction exists: $z \mapsto (G, F)(z)$)

$\boxed{\text{CHECK 2}}$ The above spans commute (this should be similar to what was done in class — and hence fairly straightforward.)