:B/2003/THU

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) ⓐ ~~We~~ Note that $U_q(\mathfrak{g})$-modules ~~are~~ have dim. indep of q (eg. fin dim $V_q(\lambda)$), though their q-dim may depend on q

This ~~old~~ phenomenon of "q-invariance" is not quite reflected in our Hall-module from last time. Roughly speaking,

$\mathcal{B}(W)$ given $W \in$ Fin dim $\mathrm{Rep}_{\mathbb{F}_q}(A_2)$,

was the category of all $V_1 \hookrightarrow V_2 \hookrightarrow W$

But now, isomorphisms in $\mathcal{B}(W)_0$ are

$$\begin{tikzcd}
V_1 \arrow[r, hook] \arrow[d, "?"] & V_2 \arrow[r, hook, "\alpha"] \arrow[d, "?"] & W \arrow[d, "1"] \\
V_1' \arrow[r, hook] & V_2' \arrow[r, hook, "\beta"] & W
\end{tikzcd}$$

So we must have $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$ to even have this morphism!

~~So, β really must have the same~~

But even then, we have many subspaces in W, depending on q. ~~So then~~

So, # isoclasses of indecomp.'s depends on q

ie dim(Hall Module) depends on q

ⓑ To amend this, we now construct a "nicer" Hall-module.

We start by replacing $\mathcal{B}(W)$ by

$$\mathrm{MON}(\mathcal{A})_0 := [(V \hookrightarrow W),\ V, W \in \mathcal{A}]_0$$

($\mathcal{A}$ was $\mathrm{Rep}_{\mathbb{F}_q}(A_2)$).

We will now construct the underlying groupoid

(whose degroupoidification is the "nice" Hall-module).

It's $(\mathrm{MON}_{\mathrm{CONST}}(\mathcal{A}))_0$ = (monomorphisms over constant rep's)

Defn A constant rep. of a quiver is a vector space X

(to be thought of, via: vs @ every vertex is X
every map ("edge") is id_X)

i.e. objects in $\mathrm{MON}_{\mathrm{CONST}}(\mathcal{A})_0$ are ~~$R_1 \hookrightarrow R_2$~~ $V_1 \hookrightarrow V_2 \hookrightarrow W$

$$\begin{tikzcd} = & V_1 \arrow[r, hook] \arrow[d, hook] & V_2 \arrow[d, hook] \\ & W \arrow[r, "1"] & W \end{tikzcd}$$

We have the "CONSTANT" functor, sending $\downarrow$ to W.

On the other hand, given any Young diagram D, one has the notion of D-flags on a vector space X and the forgetful functor, ~~that~~ takes this to X.

ⓒ We can now construct the ~~p~~ weak/homotopy pullback

$$\begin{tikzcd}
& \mathbb{P} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & \\
\mathrm{MON}_{\mathrm{CONST}}(\mathcal{A})_0 \arrow[dr, "\text{``CONSTANT''}"'] & & {[\mathbb{F}_q\text{-V-SP. W/ D-FLAG}]_0} \arrow[dl, "\text{FORGET}"] \\
& (\text{VEC SP.}_{\mathbb{F}_q})_0 &
\end{tikzcd}$$

Thus, essentially, $\mathbb{P} = \left(\begin{array}{c}\text{Vec. Spaces equipped with a}\\ \text{pair of structures}\end{array}\right)_0$

: ⓐ "quiver-rep embedded in it"
: ⓑ a D-flag on it.

ⓓ So, we're really trying to compute the "moral" & "homotopy fiber product" because these two properties/structures are independent.

(Recall how the fiber product works:

$$\begin{tikzcd}
& T\times_B T' \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"] & \\
T \arrow[dr, "\pi"'] & & T' \arrow[dl, "\pi'"] \\
& B &
\end{tikzcd}$$

$$T\times_B T' = \coprod_{b\in B} \pi^{-1}(b)\times(\pi')^{-1}(b)$$
)

But if we work over/with groupoids, then we are allowed to have

$F : G_1 \to G_2$

$R \to S$ OR $R \to S' \xrightarrow{\sim} S$

So, the fiber should be replaced in this setting by the "homotopy/moral" fiber, ie

$$F^{-1}(S) = \{(R, S' \xrightarrow[\varphi]{\sim} S, \varphi) : F(R) = S'\}$$

And weak pullbacks are also called homotopy pullbacks

= homotopy (fiber product)

= (homotopy fiber) product !

ⓐ There are several things to do now:

(i) Give an example
(ii) Counting the isoclasses (are there only fin many!)
(iii) Defining the groupoidification of the action map.

~~(iv) Not [illegible]~~:

$$\begin{tikzcd}
\mathcal{P} & & \mathcal{A} \\
& SES(\mathcal{P}, \mathcal{A}) \arrow[ul] \arrow[ur] \arrow[d] & \\
& \mathcal{P} &
\end{tikzcd}$$

Let's do (iii) first:

Where

$SES(\mathcal{P},\mathcal{A})$ is given as

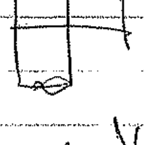
all
$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook] \arrow[d, hook] & C \arrow[r] \arrow[d, hook] & D \arrow[r, two heads] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & G \arrow[r, two heads] \arrow[dd] & 0 \\
 & X \arrow[r, "1"'] \arrow[dd] & X \arrow[dd] & & \\
 & & & \in \mathcal{A} & \\
 & \in \mathcal{P} & \in \mathcal{P} & &
\end{tikzcd}$$

, so they are ALSO assumed to have the same D-flag on them

Ⓐ Also note that comparing two types of flags, is exactly what

(i) is being done in the groupoid $\mathcal{P}$!

(ii) we also did last quarter, while to look at Hecke operators!

Ⓒ We consider an example: D = 

$A_2 \leftrightarrow SL_3 \rightarrow$ Vect $V \supseteq \boxed{V_2 \supseteq V_1}$

$\boxed{V \supseteq \mathbb{C}}$

So, vect space w/ D-flag is 2-dim $\subseteq$ 3-dim

$\boxed{V_1, V_2 \subset}$

Now Combine the 2 structures, and check/compare

Here's the comparison table:

$V \supseteq B \supseteq A$ $\qquad\qquad$ $\underset{\dim = 3}{V} \supseteq \underset{\dim = 2}{C}$

dim A	dim B	# isoclasses	# isos
0	0	$A = B \subseteq C \subseteq V$	1
0	1	$\dim(\underset{\overset{\cup\mid}{A}}{B} \cap C) =$ 0 or 1	2
0	2	$\dim B \cap C =$ 1 or 2	2
0	3	$A \subseteq C \subseteq B = V$	1
1	1	$\dim B \cap C =$ 0 or 1	2
1	2	$\dim B \cap C =$ 1 or 2. If 1, $\dim A \cap C =$ 0 or 1	3
1	3	$\dim A \cap C =$ 0 or 1	2
2	2	$\dim B \cap C =$ 1 or 2	2
2	3	$\dim A \cap C =$ 1 or 2	2
3	3	$C \subseteq A = B = V$	1

TOTAL = DIM(MODULE) $= \boxed{18}$

Qn (Why) is this a module for $U_q(\mathfrak{g})$? All we know is that it's a module over Hall alg, & $U_q(\mathfrak{n}^+)$.