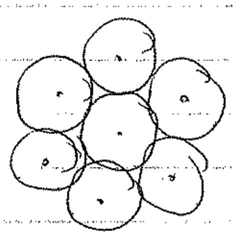
EB | 2008

John Baez – Kostant and E_8

(a) The Spin Group (= double cover) of the 8-dimensional rotations has a 3-fold symmetry called triality, which makes octonions show up, as well as the largest exceptional Lie group E_8.

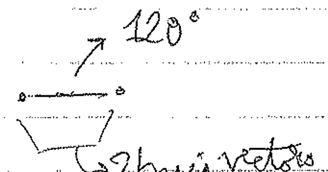
We'll talk about E_8, to provide some background for Kostant's talk later today on Garrett Lisi's use of E_8 in a theory of everything.

(b) Let's start ~~with~~ with the penny-packing problem, in which the best solution in $\mathbb{R}^2$ is the hexagonal arrangement:



ie it leads to a certain lattice $L \subseteq \mathbb{R}^2$.

This is called the A_2-lattice



and is an example of a simply laced Dynkin diagram.

(c) Packing oranges leads to a lattice $L \subseteq \mathbb{R}^3$ called A_3.

The corresponding simply-laced Dynkin diagram is $\underset{e_1}{\bullet} \overset{120°}{—} \underset{e_2}{\bullet} \overset{120°}{—} \underset{e_3}{\bullet}$ and $\angle(e_1, e_3) = 90°$

(d) ~~And~~ we can continue this way. In fact, one realization of the lattice for any A_n:

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \circ - \cdots - \underset{\alpha_n}{\circ}$$

<u>Define</u> $\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_{n+1}$ to be the standard orthonormal basis of $\mathbb{R}^{n+1}$, and define

$$\alpha_i := \underset{\sim}{e}_i - \underset{\sim}{e}_{i+1}.$$

<u>Check</u> that this works!

(2) (a) How did this lattice just show up? Not magically; in fact, there is a recipe to compute the lattice, given any compact simple Lie group G!

This is done via the <u>Lie algebra</u> $\mathfrak{g} = \mathrm{Lie}(G)$.

(b) So, here's how. Let's take the example of A_2: $\circ - \circ$

$SU(3)$ is a compact simple Lie gp $\subseteq SL(3, \mathbb{C})$

There is a "<u>maximal torus</u>"

$$T^2 \cong \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} : |\alpha| = |\beta| = |\gamma| = 1 \text{ and } \alpha\beta\gamma = 1 \right\}$$

$$\cap$$

$$SU(3)$$

(c) Now one looks at $\mathfrak{g} = \mathrm{Lie}(G)$: here, $\mathfrak{su}(3) \xrightarrow{\exp} SU(3)$

and the analog of the maximal torus T^2 is $\mathfrak{h} \subseteq \mathfrak{su}(3)$

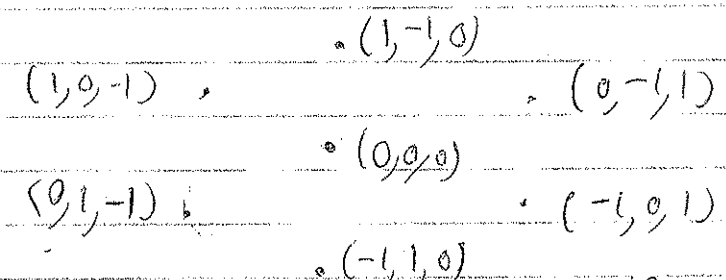
given by $\mathfrak{h} = \{X \in su(3) : \exp(tX) \in T^2 \ \forall t \in \mathbb{R}\}$

So, $\mathfrak{h} = \left\{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} : \begin{array}{l} a,b,c \in \mathbb{R}, \\ a+b+c = 0! \end{array} \ (\text{not any} \in 2\pi\mathbb{Z}) \right\}$

The kernel of $\exp : \mathfrak{h} \to T^2$ is a <u>lattice</u>

$$L = \left\{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} : \begin{array}{l} a+b+c = 0 \\ a,b,c \in 2\pi\mathbb{Z} \end{array} \right\}$$

and this is the lattice above! In fact, let's "draw" it:



and this <u>is</u> ~~the~~ the penny-packing lattice, A_2

(d) In general, it is an amazing and deep result in Lie theory, that just knowing this lattice, gives you the entire Lie group (almost; certainly the Lie algebra) itself!

(e) Similarly, $SU(n)$ has a maximal torus

$$T^{n-1} = \left\{ \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} : \begin{array}{l} |\alpha_i| = 1, \\ \alpha_i \in \mathbb{C} \end{array}, \ \prod_i \alpha_i = 1 \right\}$$

with Lie algebra $\mathfrak{t}^{n-1} = \left\{ \begin{pmatrix} ia_1 & & 0 \\ & \ddots & \\ 0 & & ia_n \end{pmatrix} : \begin{matrix} a_j \in \mathbb{R} \\ \sum_j a_j = 0 \end{matrix} \right\}$

$$\mathfrak{t}^{n-1} \cong \mathbb{R}^{n-1}$$

The kernel of the exp map is $L = \left\{ \begin{pmatrix} 2\pi i a_1 & & 0 \\ & \ddots & \\ 0 & & 2\pi i a_n \end{pmatrix} : \begin{matrix} \sum a_i = 0 \\ a_i \in \mathbb{Z} \end{matrix} \right\}$

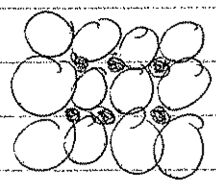
This is the A_n-lattice, and the suitable basis for the Dynkin diagram was mentioned above.

— x —

(3) In 3-dim, it was only proved <u>very</u> recently, that the A_3-lattice is the best way to pack oranges/balls.

(a) In 4-dim, the <u>densest known</u> packing is the D_4-lattice — not the A_4!

(b) How to consider the D_4-lattice? Well, consider the <u>cubical</u>/rectangular lattice in $\mathbb{R}^n$, eg in $n=2$,



, and then stick in "smaller" circles • in between the ○'s.

For $n = 2, 3$, note that (1) the radii of the • are < ○
(2) centers of ○'s can be denoted by integers; then the centers of • have <u>all</u> coord. $\frac{1}{2}$-integers.

For $n = 4$, (2) still holds, but — bingo! — the radius of • turns out to <u>equal</u> the radius of ○!

So, we directly get a packing with double the density to the rectangular packing — and this is the D_4 lattice! Explicitly,

$$D_4 = \left\{ (a,b,c,d) \in \mathbb{R}^4 : \begin{array}{l} a,b,c,d \in \mathbb{Z} \;\& \\ a,b,c,d \in \mathbb{Z} + \frac{1}{2} \end{array} \right\}$$

(c) Which vectors are closest to the origin in D_4?

$(\pm \underset{\sim}{e_i}) \longrightarrow 8$ of ~~them~~ $\longrightarrow$ make a "hyper-octahedron"

$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \rightarrow 16$ of these $\rightarrow$ make a "hyper cube"

Together, there are 24 of these, ie 24 balls are touching the "origin-al" ball.

↓

make a "4-24-cell" $\rightarrow$ = union of two 4-platonic solids

(24 ~~edges~~ vertices, 24 faces)

↓

Self dual.

(d) (In general, there are only 3 platonic solids in $\mathbb{R}^n$ for $n > 4$:

hypercube
hyper simplex $\rightarrow$ self dual
hyper octahedron
dual

In $n=3$, 5 solids (dodeca— & icosa—)
In $n=4$, 6 solids (hyper's of all 3-d's, and the self-dual 24-cell)

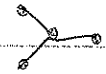
④ ⓐ In dim 5, D_5 is the densest.

$\boxed{\text{Note}}$ We are now only restricting ourselves of <u>lattice packings</u>, in general we don't know, but we ~~most~~ know for lattices, what the densest is, up to <u>23 dimensions</u>.

ⓑ In general, D_n has Dynkin diagram  D_n

with $D_2 \cong A_1 \times A_1$:

$D_3 \cong A_2$ 

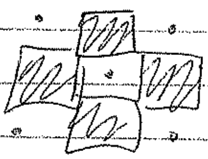
and D_4 =  → has a 3-fold symmetry of rotations in $\mathbb{R}^8$!

~~#2~~ A4

(So, we get 3 reps of $SO(8)$ (& $Spin(8)$) that are "<u>similar</u>" ....) ← (This is called <u>triality</u>.)

D_n corresponds to the Lie group $SO(2n)$.

ⓒ Another way to define D_n is to ~~differ~~ use a different coordinate system as a "<u>checkerboard lattice</u>" —

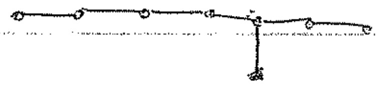
 etc. So, $D_n = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \sum a_i \in 2\mathbb{Z}\}$

Thus, $D_2 \cong \mathbb{Z}_2$ as lattices $\subseteq$ inner product space
$\cong A_1 \times A_1$

$D_3 \cong A_3$, $D_4 \cong$ old D_4 $\not\cong A_4$

(5)
(a) In 8-dimensions, you could try D_8, but the gaps between spheres are big enough to slip in more spheres of the same size! This gives E_8. So,

$$E_8 = \left\{ (a_1, \ldots, a_8) : \sum_i a_i \in 2\mathbb{Z}, \text{ and all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \tfrac{1}{2} \right\}$$

and it has Dynkin diagram 

(b) This can be checked from the construction of the vectors closest to the origin:

$(1, -1, 0, \ldots, 0)$ or in general, 2 coord's (± 1)
6 coord's (0)

$$\Rightarrow \binom{8}{2} \cdot 2^2 = 112$$

and $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right) \rightarrow 2^8/2^1 = 128$ → 1 constraint, add/subtract the last $\frac{1}{2}$ to get in $\underline{2\mathbb{Z}}$.

So, there are $112 + 128 = 240$ vectors nearest to the origin.

(c) Note: In A_n, the vectors closest to the origin are the $(e_i - e_j)\ \forall i \neq j$.

This is a general phenomenon:

(d) In general, (i) the lattice = kernel $(\mathfrak{h} \xrightarrow{\exp} T)$ is isomorphic, as a lattice inside an inner product space to the root lattice (or weight lattice).

(ii) The no. of dots in the Dynkin diagram $= \dim(\mathfrak{h}) = rk(\mathfrak{g})$

(iii) ~~The~~ vectors closest to the origin form the root system.

(e) So, $\dim(\text{Lie gp}) = (ii) + (iii)$, eg. $\dim(SU_3) = 2+6 = 8$

and $\dim(E_8) = 8 + 240 = \boxed{248}$

(f) Remark It is possible to embed the lattices

$A_2 \,\&\, A_1 \times A_1 \subseteq \mathbb{C} \cong \mathbb{R}^2$.

$D_4 \subseteq \mathbb{H} \cong \mathbb{R}^4$

and $E_8 \subseteq \mathbb{O} = \text{octonions} \cong \mathbb{R}^8$

So that ~~the~~ the lattices are closed under both addition and multiplication.

(PTO)

(g) Note on part (d): For any Lie group G as above (simple), G has various analogues — from its simply conn. universal cover $\tilde{G}$, to its adjoint form $G/Z(G)$.

The kernel from ~~this~~ $\tilde{G}$ is the smallest lattice ~~containing the~~ and the kernel from $G/Z(G)$ is the largest

These are the root lattice and weight lattice resp.

Moreover the fundamental groups of the spaces are "inversely proportional" in size to the spaces,

so, $\pi_1(\tilde{G}) = 1$ and

$\pi_1(G/Z(G)) \cong$ wt. lattice / root lattice

which is why it is sometimes called the fundamental group of $\mathfrak{g}$ too!