Jim Dolan

(1) (a) Last time, we worked with $\bullet\!-\!\bullet$ $A_2 = sl_3$, and came up with a Hall-module of dim $= 18$.

Why 18? Here's how to read off dimensions ~~from the~~ of the Hall modules, associated to a particular diagram.

Each Row of size $\underbrace{\square\square\square\cdots\square}_{d}$ corresponds to $\mathrm{Sym}^d \square$

And Columns mean

$$\begin{array}{l}\boxed{}\\ \boxed{}\\ \boxed{}\end{array} = \boxed{} \otimes \square \otimes \square$$

So, it all depends solely on the initial value / data (of $\square$)!

(b) Now for our purpose, $\underset{3}{\square} = \mathbb{C}^3 = V_3 =$ ~~the fundamental~~ tautological rep of sl_3 $\quad T = T_3$

and then $\begin{array}{l}\square\square\\ \square\end{array} = V_3 \otimes \mathrm{Sym}^2 V_3 \to$ has dim $3 \times \binom{4}{2} = 18$

$$\left[\dim\left(\underbrace{\square\square\square\cdots\square}_{d}\right) = \binom{d+2}{2} \text{ in general}\right]$$

(c) Let's try to see how an 18-dim module $= \mathbb{C}^3 \otimes \mathbb{C}^6$.

Recall again: we have to consider the interplay between

$V_3 \supseteq X_2 \supseteq X_1$ and $V_3 \supseteq V_2 =$ plane.

$\dim X_1$	$\dim X_2$	$(\dim(V_2 \cap X_1), \dim V_2 \cap X_2)$			#
0	0	$(0,0)$			1
0	1	$(0,0)$	$(0,1)$		2
0	2	$(0,1)$	$(0,2)$		2
0	3	$(0,2)$			1
1	1	$(0,0)$	$(1,1)$		2
1	2	$(0,1)$	$(1,1)$	$(1,2)$	3
1	3	$(0,2)$	$(1,2)$		2
2	2	$(1,1)$	$(2,2)$		2
2	3	$(1,2)$	$(2,2)$		2
3	3	$(2,2)$			1

How to see 3×6? Each of the following occurs thrice!

$(0,0)$, $(0,1)$, $(0,2)$, $(1,1)$, $(1,2)$, $(2,2)$

which is really like a basis of $\mathrm{Sym}^2(V_3)$!

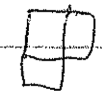
(d) We've thus "obtained" modules over the quantum group.

How does one "obtain" morphism-spaces between these modules?

To do this, note also that Young diagrams represent "functorial operations on reps" → called Schur functors.

E.g. All our diagrams above are the Schur functors ~~of~~

$\mathrm{Sym}^n(-)$ & $- \otimes -$ applied to V_3.

i.e.  $(V_3) = \text{---}$

(e) Now, we won't just groupoidify the Hom-spaces between the values of the Schur functors — we'll groupoidify the Hom spaces between the Schur functors themselves!

i.e. we'll be looking at natural transformations.

(f) The Theory of Schur functors is related to Schur–Weyl duality,

which relates the Rep Theory of $GL(N)$ with that of $K!$

via the rep

$$\underbrace{T_N \otimes \cdots \otimes T_N}_{K\text{-times}}$$

as a simultaneous rep of $GL(N)$ and $K!$

(because both embed into the wreath products $GL_N \wr S_K$ ("$K!$"))

And now, Schur–Weyl duality allows us to transfer Young diagrams from the $K!$-setting, to the $GL(K)$-setting, as above.