FEB/2008/TUE II

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1) Recall: We have "Fock space" $k[x_1, \ldots, x_n]$ with annihilation and creation operators $a_i, a_i^* : k[x_1, \ldots, x_n] \circlearrowleft$
where $a_i = \frac{\partial}{\partial x_i}$, $a_i^* = m_{x_i}$

(a) These generate an algebra of linear transformations of Fock space, called the (n^{th}) Weyl algebra W_n. Abstractly, it's the algebra generated by elements $\{a_i, a_i^* : 1 \leq i \leq n\}$ modulo the relations

$$a_i a_j = a_j a_i \qquad \forall i, j$$
$$a_i^* a_j^* = a_j^* a_i^*$$
$$[a_i, a_j^*] = \delta_{ij}$$

What is nontrivial is that these are all the relations "satisfied in W_n". (To check this, it's enough to show that for every polynomial differential operator D, we can find a polynomial not killed by D.)

(b) W_n has lots of interesting Lie subalgebras, eg

$\mathfrak{gl}(n) = Mat_{n\times n}(k)$. How do we see this?

Well, $\mathfrak{gl}(n)$ has a basis $\{e_{ij} : 1 \leq i,j \leq n\}$ of elementary matrices

$$((e_{ij}))_{kl} = \delta_{ik} \delta_{jl} \quad \left(\begin{array}{l} 1 \text{ in } i^{th} \text{ row } \& \ j^{th} \text{ column} \\ 0 \text{ otherwise.} \end{array} \right)$$

We'll present a set-map $\rho : \{e_{ij}\} \to W_n$ that extends by linearity to a ~~Lie~~ faithful Lie alg. rep.

Define $\rho(e_{ij}) = a_i^* a_j$. We've seen 2 weeks ago in HW, that

$$[a_i^* a_j, a_k^* a_l] = \delta_{jk}\, a_i^* a_l - \delta_{il}\, a_k^* a_j$$

and all we need to check is that

$[e_{ij}, e_{kl}]$ has the same form: $\delta_{jk}\, e_{il} - \delta_{il}\, e_{kj}$

ⓒ A More Conceptual / Illuminating Approach, at least for $k = \mathbb{R}$ or $\mathbb{C}$,

Here we have the Lie group $GL_n(k)$ of $n \times n$ invertible matrices, which acts on $\mathbb{R}^n$ as linear (invertible) transformations.

These are all (C^∞) diffeomorphisms (smooth maps w/ smooth inverses)

ⓓ [Indeed, given any Lie group G acting as diffeomorphisms of a manifold

$$R : G \to Diff(M)$$

(a smooth map of smooth groups)

we get a Lie alg. hom.

$$\rho = dR : \mathfrak{g} = Lie(G) \to Vect(M) = Lie(Diff(M)) \;]$$

ⓒ (cont.) So, the action R of GL_n (on $\mathbb{R}^n$) as diffeomorphisms, gives

$$\rho = dR : \mathfrak{gl}(n) \to Vect(\mathbb{R}^n)$$

And in fact, $\rho \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \sum_{i,j=1}^{n} a_{ij}\, x_i \frac{\partial}{\partial x_j}$

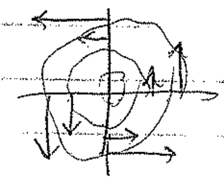
ⓔ Example: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponds to $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$

$e^{t\begin{pmatrix}0&1\\-1&0\end{pmatrix}} \in GL_2$ ($t \in \mathbb{R}$) now corresponds to e^{it}, which is rotation by the angle t.

Moreover, the flow lines of the vector field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathrm{Vect}(\mathbb{R}^2)$

are concentric circles, i.e.

generates rotations



So we're getting $\rho : \mathfrak{gl}(n) \hookrightarrow \mathrm{Vect}(\mathbb{R}^n)$ whose image is all the linear vect. fields.

But W_n is the algebra of polynomial coefficient differential operators e.g. $x_1^3 x_2 \frac{\partial^2}{\partial x_1 \partial x_3}$. So we get

$$\rho : \mathfrak{gl}(n) \longrightarrow W_n .$$

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(2) (a) We can groupoidify much of this story. We can groupoidify Fock space and get $\mathrm{FinSet}_0^n \to$

$$H_0(\mathrm{FinSet}_0^n) = k[x_1, \dots, x_n]$$

We can groupoidify a_i & a_i^*, making them into spans of groupoids:

$$\boxed{A_i^*} \quad \begin{tikzcd} & \mathrm{FinSet}_0^n \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\underline{x} \mapsto \underline{x} + \underline{e}_i"] & \\ \mathrm{FinSet}_0^n & & \mathrm{FinSet}_0^n \end{tikzcd} \qquad \boxed{A_i} = (A_i^*)^* .$$

(b) Next, we would like to groupoidify the representation ρ of $gl(n)$ on Fock space, ie $\rho : gl(n) \to W_n \subseteq \mathrm{End}_k(k[\underline{x}])$

Problem: Groupoidify these vector spaces: $\downarrow$, $\downarrow$, $\downarrow$, and the map ρ ~~...~~ into a span of groupoids between them, so that we still get a "Lie alg. hom."

(c) It's easier to groupoidify individual operators: $\rho(e_{ij}) \in W_n \subseteq \mathrm{End}(k[\underline{x}])$

So we should look for spans S_{ij} 'FinSets^n' ?

What are these? Well, $\rho(e_{ij}) = a_i^* a_j$, so the obvious candidate for the span is

$$A_i^* A_j = E_{ij}$$

With luck, this will "work" — ie, we have $[\rho(x), \rho(y)] = \rho[x,y]$

and as seen earlier, we have "$-$" in some of the terms. So we must push them on the other side...!

ie we have $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$

ie $\boxed{e_{ij} e_{kl} + \delta_{il} e_{kj} = e_{kl} e_{ij} + \delta_{jk} e_{il}}$

So - we are to show

$$\boxed{E_{ij} E_{kl} + \delta_{il} E_{kj} \cong E_{kl} E_{ij} + \delta_{jk} E_{il}}$$

We can check this — JB thinks it's true.

(d) Moral: Each elementary matrix $e_{ij} \in gl(n)$ acts as a transmutation operator $a_i^* a_j$ on Fock space —

which turns a particle of type j into a particle of type i. And we can groupoidify all this.

(e) But we'd like to groupoidify all of $gl(n)$. Or at least "half" of $gl(n)$, ie the "Borel" Lie subalgebra

eg $$\boxed{\mathfrak{b} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \text{ for } gl(n)}$$ (upper triangular) = maximal solvable Lie subalg

Why? It turns out that $\mathfrak{b}$ here, is related to the "A_n-quiver" $(\cdot \to \cdot \to \cdots \to \cdot)$ (n dots)

(f) ~~How~~? There's an easy relation and a deeper one

$\boxed{\text{Easy relation}}$: A quiver is a category freely generated by a directed graph Q:

(ie we also have the composite / concatenated paths)

For $Q = A_n$ this category has objects corresponding ~~to~~ the dots $i = 1, 2, \cdots, n$, and morphisms $e_{ij}: j \to i \quad \forall i \geq j$.

($i = j$ gives identity morphism e_{ii} @ i in our category.)

Now $\mathcal{C}_{A_n}$ has all morphisms in ~~[illegible]~~ $(\mathrm{Mor}_{\mathcal{C}_{A_n}})$

So we now have three constructions:

(i) $\mathfrak{b}$ which is actually an associative algebra

(ii) the path algebra of a quiver (Q).

(iii) the Category-algebra $= \operatorname{Free}(\operatorname{Mor}(\mathcal{C}_{A_n}))$

with mult. defined by $\begin{cases} \text{composing paths} \\ 0 \text{ otherwise} \end{cases}$

$\boxed{\text{FACT}}$: (i) $\cong$ (ii)(A_n) $\cong$ (iii) as associative algebras!

(8) Note: There's still the problem about $sl(n) \leftrightarrow A_{n-1}$ / $gl(n) \leftrightarrow A_n$ / etc.