

4/MAR

John Baez - Hall Algebras & Quantum Groups

1 a

Given a simple complex Lie algebra \mathfrak{g} , there are various Lie groups associated with it, going from a simply connected one, to the centerless one (of "adjoint type"?) \rightarrow let us call the latter $g \cdot G$.

The only normal subgroups of these Lie groups are all discrete, and \mathbb{Z} , G has no normal subgroup except $\mathbb{1}$, G . Thus, G really is a simple Lie group.

(The reason all the above Lie groups are called "simple" too, is by \downarrow , these normal subgroups won't show up when we pass to the Lie algebra \mathfrak{g} .)

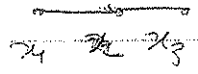
b

Pick a maximal abelian subgroup $H \subseteq G$ (any two are conjugate under G , hence isomorphic). This has a Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ and \mathfrak{h} has a God-given inner product on it (the Killing form) and \mathfrak{h} has a lattice $L \subseteq \mathfrak{h}$, given by:

$$L = \ker(\exp: \mathfrak{h} \rightarrow H)$$

c

Sometimes L has a basis of vectors that are all of the same length and at angles 90° & 120° . Then you can draw a "simply-laced Dynkin diagram", eg



An edge between dots corresponding to basis vectors x_i & x_j if they're at a 120° angle; no edge otherwise.

Theorem

The only diagrams you get this way are A_n, D_n, E_6, E_7, E_8

Thus, we have arrived at this list (which we also get in the quiver-setting) via a completely different way!

d) eg. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

Here, $\mathfrak{h} = \{ \text{diagonal matrices with trace } 0 \}$

and $\exp: \mathfrak{h} \rightarrow H$ sends $\text{diag}(\alpha_1, \dots, \alpha_n) \mapsto (e^{\alpha_1}, \dots, e^{\alpha_n})$.

So $L = \left\{ \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} : \sum_{i=1}^n \alpha_i = 0, \alpha_i \in 2\pi\mathbb{Z} \right\}$

If $\mathfrak{g} = \mathfrak{sl}(3)$ then L has a basis $\begin{pmatrix} 2\pi & 0 & \\ & -2\pi & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 0 \\ & 2\pi & \\ & & -2\pi \end{pmatrix}$

which are at a 120° angle w.r.t the Killing/Trace forms

$\langle x, y \rangle := c \cdot \text{Tr}(xy)$ for a fixed $c \in \mathbb{R}^+$

and the lattice L is the usual "hexagonal/ Δ^2 lattice".

e) In this general situation, we have a vector space decomposition:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Here, \mathfrak{h} is as above (and any two of these are \mathbb{C} -conjugate) and called the "Cartan subalgebra" of \mathfrak{g} .

\mathfrak{n}_\pm are maximal nilpotent subalgebras.

[eg. upper Δ^2 (strictly) matrices in $\mathfrak{sl}(n, \mathbb{C})$.

In fact, \downarrow is the example, because every nilpotent algebra embeds into strictly upper Δ^2 matrices on some vector space.

Example

$$\mathfrak{g} = \underbrace{\left\{ \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \right\}}_{\mathfrak{n}_-} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \right\}}_{\mathfrak{h}} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \right\}}_{\mathfrak{n}_+}$$

f) In this example — as in general! — $[b, \mathfrak{h}_\pm] \subseteq \mathfrak{h}_\pm$
(of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$) (In fact, $[b, \mathfrak{h}_\pm] = \mathfrak{h}_\pm$)

Moreover, \mathfrak{h}_+ has a nice basis of elementary matrices

$$\{E_{ij} : 1 \leq i < j \leq n\}$$

They are called root vectors, and analogues exist for all semisimple \mathfrak{g} .

—X—

2) Now recall the Ringel-Hall Theorem: If $\mathcal{Q} = A_m, D_n, E_6, E_7, E_8$

a)

then for any prime power, we have an isomorphism of assoc. algebras:

$$\text{Hall}(\text{Rep}_q(\mathcal{Q})) \cong U_q(\mathfrak{h}_+)$$

where $\text{Rep}_q(\mathcal{Q})$ is the abelian category $\text{hom}(\mathcal{Q}, \text{FinVect}_q)$.

(reps / functors as objects, nat. trans / maps b/w reps, as morphisms)

and $U_q(\mathfrak{h}_+)$ is the " q -deformed universal enveloping algebra (UEA) of \mathfrak{h}_+ "



b)

This is part of a bigger algebra called the "quantum group".

But what's $U_q(\mathfrak{h}_+)$? We won't address this today. However, we first ask: what's $U(\mathfrak{h}_+)$?

Ans: "U" means "universal enveloping algebra" — it's the (key) trick for turning Lie algebras into associative algebras.

c) To see this, note that there's a "forgetful" functor

$$F: \text{Assoc Alg}_k \longrightarrow \text{Lie Alg}_k \quad (\text{for a field } k)$$

where given $A \in \text{Assoc Alg}$, $F(A)$ is the same underlying vector space, equipped with a Lie bracket:

$$[a, b] := ab - ba.$$

d) This functor has a left adjoint U :

$$\left(\begin{array}{c} \text{Assoc Alg} \\ \uparrow U \quad \downarrow F \\ \text{Lie Alg} \end{array} \right)$$

(Oddly enough, in category theory, one often uses U for "underlying" = forgetful, and F is called "free", and they are used for exactly the reverse functors to this case!)

Reminder: By the adjointness, we mean:

given $A \in \text{Assoc Alg}$, $L \in \text{Lie Alg}$,

$$\text{hom}_{\text{Lie Alg}}(L, F(A)) \cong \text{hom}_{\text{Assoc Alg}}(U(L), A)$$

e) Concretely, $U(L)$ is the assoc. alg-freely gen'd by ets. of L , modulo the relations

$$\left. \begin{array}{l} \alpha(xy + \beta z) = \alpha xy + \beta \alpha z \\ (\alpha x + \beta y)z = \alpha xz + \beta yz \\ xy - yx = [x, y] \end{array} \right\} \forall x, y, z \in \mathfrak{g}, \alpha, \beta \in k.$$

~~→ the Jacobi identity~~

Then given a Lie alg. hom. $f: L \rightarrow FA$, it extends uniquely to an assoc. alg. hom. $\tilde{f}: UL \rightarrow A$ (b/c of the third relation).

This can be rephrased as a "universal property":

$$\begin{array}{ccc} UL & \xrightarrow{\exists! \tilde{f}} & A \\ \uparrow & & \uparrow \cong \\ L & \xrightarrow{\forall f} & FA \end{array}$$

—X—

③ We now apply all this to work towards $U(\mathfrak{m}_+)$.

① Example: $L = \mathfrak{m}_+ \subseteq \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) \cong \mathfrak{gl}(n, \mathbb{C})$
 $A = \text{End}(\mathbb{C}^n)$. $FA = \mathfrak{gl}(n, \mathbb{C})$
 $UL = U(\mathfrak{m}_+) (=?)$

(Note: ^{strictly} Upper Δ^n matrices are closed under matrix mult.
 But this is NOT the UEA of \mathfrak{m}_+ !

② Now, there's an obvious inclusion isomorphism (of Lie algebras)

$$f: L \hookrightarrow \mathfrak{sl}(n, \mathbb{C}) \hookrightarrow \mathfrak{gl}(n, \mathbb{C}) = FA$$

③ This yields \Downarrow an assoc. algebra homomorphism

$$\tilde{f}: UL \rightarrow A, \text{ i.e. } \tilde{f}: U(\mathfrak{m}_+) \rightarrow \text{End}(\mathbb{C}^n)$$

where \tilde{f} sends $x \in \mathfrak{m}_+$ to the element of $\text{End}(\mathbb{C}^n)$ that it secretly is!

④ We also have an assoc. alg. hom.: $\text{End}(\mathbb{C}^n) \rightarrow \text{End}(\text{Sym}(\mathbb{C}^n))$
 where $\text{Sym}(\mathbb{C}^n)$ is the comm. assoc. alg. gen'd by \mathbb{C}^n , i.e. $\mathbb{C}[x_1, \dots, x_n] = \text{polynomials in } n \text{ variables}$.

It's also ~~the~~ the Fock Space (space-Hilbert space) we've been talking about, all along!

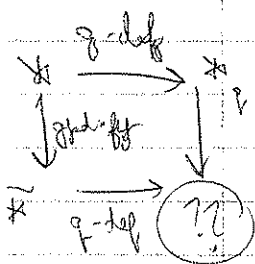
Composing, we get $\text{alg map } U_{\mathfrak{h}_+} \rightarrow \text{End}(\mathbb{C}^n) \rightarrow \text{End}(\text{Sym } \mathbb{C}^n)$

which is a restriction of $U(\mathfrak{h}_+) \rightarrow \text{End}(\text{Sym } \mathbb{C}^n)$
 that we've discussed earlier!
 eg the e_{ij} 's are sent to the corresponding "transmutation operator" $a_i^* a_j$ (on Fock space)

(d) This is the thing we'd like to categorify (we've done that via spans of groupoids - groupoidified transmutation operators $A_i^* A_j$ for all i, j)

- but also q-deform! (But here we'll be stuck with $i < j$, since we're using $U(\mathfrak{h}_+)$ instead of $U(\mathfrak{g})$.)

Note



The reason we don't ~~do~~ work with $U(\mathfrak{g})$, though it has been both q-deformed and groupoidified, is because we cannot seem to be able to complete the "square"!

(e) Now, what's $U_q \mathfrak{h}_+$? As a vector space, it's $\cong U\mathfrak{h}_+$, but it has a different product \cdot_q ($q \in \mathbb{C}^*$), with $x \cdot_q y \rightarrow xy$ as $q \rightarrow 1$

We say that $U_q(\mathfrak{h}_+)$ is the "classical limit" for $U_q(\mathfrak{h}_+)$

(f) Next time: We'll assume: $U_q(\mathfrak{h}_+) \cong \text{Hall}(\text{Rep}_q(Q))$, and ~~to~~ ^(in a nice way)
 (i) Show that $U_q(\mathfrak{h}_+) \cong U\mathfrak{h}_+$ as vector-spaces (filtered?)
 (ii) work out product here; (iii) show: $x \cdot_q y \rightarrow xy$ as $q \rightarrow 1$