4/MAY

John Baez — Hall Alg's & Quantum Groups

(1) (a) Given a simple complex Lie algebra $\mathfrak{g}$, there are various Lie groups associated with it, going from a simply connected one, to the centerless one (of "adjoint type" (?)) → let us call this latter gp. G.

The only normal subgps of these Lie groups are all discrete, and so, G has no normal subgroup except 1_G. Thus, G really is a simple Lie group.

(The reason all the above Lie groups are called "simple" too, is: by ↓, these normal subgroups won't show up when we pass to the Lie algebra $\mathfrak{g}$.

(b) Pick a maximal abelian subgroup $H \leq G$ (any two are conjugate under G, hence isomorphic). This has a Lie algebra $\mathfrak{h} \leq \mathfrak{g}$. ~~and~~ $\mathfrak{h}$ has a God-given inner product on it (the Killing form) and $\mathfrak{h}$ has a lattice $L \leq \mathfrak{h}$, given by:

$$L = \ker(\exp : \mathfrak{h} \to H)$$

(c) Sometimes L has a basis of vectors that are all of the same length and at angles $90°$ & $120°$. Then you can draw a "simply-laced Dynkin diagram", e.g.

$$x_1 - x_2 - x_3$$

An edge between dots corresponding to basis vectors x_i & x_j if they're at a $120°$ angle; no edge otherwise.

Theorem: The only diagrams you get this way are A_n, D_n, E_6, E_7, E_8

Thus, we have arrived at this list (which we also got in the quiver-setting) ~~one~~ via a completely different way!

(d) eg. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

Here, $\mathfrak{h} = \{$diagonal matrix with trace $0\}$

and $\exp : \mathfrak{h} \to H$ sends $\mathrm{diag}(\alpha_1, \ldots, \alpha_n) \mapsto (e^{\alpha_1}, \ldots, e^{\alpha_n})$.

So $L = \left\{ \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} : \sum_{i=1}^{n} \alpha_i = 0,\ \alpha_i \in 2\pi\mathbb{Z} \right\}$

If $\mathfrak{g} = \mathfrak{sl}(3)$ then L has a basis $\begin{pmatrix} 2\pi & & 0 \\ & -2\pi & \\ 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 0 \\ & 2\pi & \\ 0 & & -2\pi \end{pmatrix}$

which are at a $120°$ angle w.r.t. the Killing / Trace Form

$\langle x, y \rangle := c \cdot \mathrm{Tr}(xy)$ for a fixed $c \in \mathbb{R}^+$; $\mathbb{C}^\times$;

and the lattice L is the usual "hexagonal / Δ^r lattice".

(e) In this general situation, we have a vector space decomposition:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Here, $\mathfrak{h}$ is as above (and any two of these are G-conjugate) and called "the"/a Cartan subalgebra of $\mathfrak{g}$

$\mathfrak{n}_\pm$ are maximal nilpotent subalgebras.

[eg. upper Δ^r (strictly) matrices in $\mathfrak{sl}(n, \mathbb{C})$.

In fact, ↓ is "the" example because every nilpotent algebra embeds into strictly upper Δ^r matrices on some vector space.]

$\boxed{\text{Example}}$

$$\mathfrak{g} = \underbrace{\left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\}}_{\mathfrak{n}_-} \oplus \underbrace{\left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}}_{\mathfrak{h}} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}}_{\mathfrak{n}_+}$$

(f) In this example — as in general! — $[\mathfrak{h}, n_\pm] \subseteq n_\pm$
($\mathfrak{g} = sl(n,\mathbb{C})$) (In fact, $[\mathfrak{h}, n_\pm] = n_\pm$)

Moreover, n_+ has a nice basis of <u>elementary matrices</u>

$$\{e_{ij} : 1 \le i < j \le n\}$$

They are called <u>root vectors</u>, and analogues exist for all semisimple $\mathfrak{g}$.

—x—

(2) Now recall the $\boxed{\text{Ringel-Hall Theorem}}$: If $Q = A_n, D_n, E_6, E_7, E_8$

(a) then for any prime power, we have an isomorphism of assoc. algebras:

$$\mathrm{Hall}(\mathrm{Rep}_q(Q)) \cong U_q(n_+).$$

where $\mathrm{Rep}_q(Q)$ is the abelian category $\mathrm{hom}(Q, \mathrm{FinVect}_q)$

(reps / functors as objects, nat. trans / maps b/w reps, as morphisms)

and $U_q(n_+)$ is the "<u>q-deformed universal enveloping algebra</u> (UEA) of n_+"

(b) This is part of a bigger algebra called the "quantum group".

But what's $U_q(n_+)$? We won't address this today. However, we first ask: What's $U(n_+)$?

<u>Ans</u>: "U" means "<u>universal enveloping algebra</u>" — it's the (key) trick for turning Lie algebras into associative algebras.

(c) To see this, note that there's a "forgetful" functor

$$F: \text{Assoc. Alg}\big|_k \longrightarrow \text{Lie Alg}\big|_k \quad \text{(for a field } k\text{)}$$

Where given $A \in \text{Assoc Alg}$, $F(A)$ is the same underlying vector space, equipped with a Lie bracket:

$$[a,b] := ab - ba.$$

(d) This functor has a left adjoint U:

$$\left(\begin{array}{c} \text{Assoc Alg} \\ U \uparrow \;\; \downarrow F \\ \text{Lie Alg} \end{array}\right)$$

(Oddly enough in category theory, one often uses U for "underlying" = forgetful, and F is called "free", and they are used for exactly the reverse functors to this case!)

Reminder: By the adjointness, we mean:

given $A \in \text{Assoc. Alg}$, $L \in \text{Lie Alg}$,

$$\hom_{\text{Lie Alg}}(L, F(A)) \cong \hom_{\text{Assoc Alg}}(U(L), A)$$

(e) Concretely, UL is the assoc. alg. freely gen'd by elts. of L, modulo the relations

$$\left.\begin{array}{rl} x(\alpha y + \beta z) &= \alpha xy + \beta xz \\ (\alpha x + \beta y)z &= \alpha xz + \beta yz \\ xy - yx &= [x,y] \end{array}\right\} \forall x,y,z \in \mathfrak{g}, \quad \alpha, \beta \in k$$

~~Jacobi identity~~

Then given a Lie alg-hom $f: L \to FA$, it extends uniquely to an assoc. alg. hom. $\tilde{f}: UL \to A$ (b/c of the third relation).

This can be rephrased as a "universal property":

$$\begin{tikzcd} UL \arrow[r, dashed, "\exists ! \tilde{f}"] & A \\ L \arrow[u] \arrow[r, "\forall f"] & FA \arrow[u, "\cong"'] \end{tikzcd}$$

—X—

(3) We now apply all this to work towards $U(n_+)$.

(a) Example: $\left.\begin{array}{l} L = n_+ \subseteq \mathfrak{g} = sl(n, \mathbb{C}) \\ A = End(\mathbb{C}^n) \end{array}\right\} \Rightarrow \begin{array}{l} FA = gl(n, \mathbb{C}) \\ UL = U(n_+) \ (=?) \end{array}$

(Note: strictly Upper Δ^r matrices are closed under matrix mult. But this is NOT the UEA of n_+!)

(b) Now, there's an obvious inclusion morphism (of Lie algebras)

$f: L \hookrightarrow sl(n, \mathbb{C}) \hookrightarrow gl(n, \mathbb{C}) = FA$.

This yields ~~F~~ an assoc. algebra homomorphism

$\tilde{f}: UL \to A$, i.e. $\tilde{f}: U(n_+) \to End(\mathbb{C}^n)$

where ~~...~~ $\tilde{f}$ sends $x \in n_+$ to the element of $End(\mathbb{C}^n)$ that it secretly is!

(c) We also have an assoc. alg. hom.: $End(\mathbb{C}^n) \to End(Sym(\mathbb{C}^n))$ where $Sym(\mathbb{C}^n)$ is the comm. assoc. alg. gen'd by $\mathbb{C}^n$, i.e. $\mathbb{C}[x_1, \ldots, x_n]$ = polynomials in n variables.

It's also ~~about~~ the Fock Space (= pre-Hilbert space) we've been talking about, all along!

Composing, we get alg map $U\mathfrak{n}_+ \longrightarrow \mathrm{End}(\mathbb{C}^n) \longrightarrow \mathrm{End}(\mathrm{Sym}\,\mathbb{C}^n)$

which is a restriction of $U(\mathfrak{gl}_n) \longrightarrow \mathrm{End}(\mathrm{Sym}\,\mathbb{C}^n)$ that we've discussed earlier!

eg the e_{ij}'s are sent to the corresponding "transmutation operator" $a_i^* a_j$ (on Fock space)

(d) This is ~~the~~ thing we'd like to categorify (we've done that via spans of groupoids — groupoidified transmutation operators $A_i^* A_j$ for all i, j.)

— but also q-defsm! (But here we'll be stuck with $i < j$, since we're using $U(\mathfrak{n}_+)$ instead of $U\mathfrak{g}$.)

$\boxed{\text{Note}}$

$$\begin{tikzcd} * \arrow[r, "q\text{-def}"] \arrow[d, "\text{grpd-ify}"] & *_q \arrow[d] \\ \tilde{*} \arrow[r, "q\text{-def}"'] & ?? \end{tikzcd}$$

The reason we don't work with $U\mathfrak{g}$, though it has been both q-deformed and groupoidified is because we cannot seem ~~to~~ be able to complete the "square"!

(e) Now, what's $U_q\mathfrak{n}_+$? As a vector space, it's $\cong U\mathfrak{n}_+$, but it has a different product $\cdot_q$ ($q \in \mathbb{C}^*$), with $\boxed{x \cdot_q y \to xy \text{ as } q \to 1}$

We say that $U(\mathfrak{n}_+)$ is the "classical limit" for $U_q(\mathfrak{n}_+)$

(f) Next time: We'll assume: $U_q(\mathfrak{n}_+) \cong \mathrm{Hall}(\mathrm{Rep}_q(Q))$, and ~~try to~~ (in a nice way)

(i) Show that $U_q(\mathfrak{n}_+) \cong U(\mathfrak{n}_+)$ as vector-spaces (filtered?)

(ii) Work out product here; (iii) Show: $x \cdot_q y \to xy$ as $q \to 1$