Geometric Representation Seminar Homework
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Homework 3

Find the unique inner product on $\mathbb{C}[z]$ *such that* m_z*, multiplication by* z*, and* $\frac{d}{dz}$*, differentiation with respect to* z*, are adjoints, i.e.*

$$\langle z \cdot z^n, z^m \rangle = \langle z^n, \frac{d}{dz} z^m \rangle$$

for all $n, m \geq 0$*, where* $||z|| = 1$.

Solution

First note that $||z|| = 1$ if and only if $||1|| = 1$, since

$$\langle z, z \rangle = \langle 1, \frac{d}{dz} z \rangle = \langle 1, 1 \rangle.$$

Now it's easy to check that all off-diagonal terms are zero, because if $n > m$, we have

$$\langle z^n, z^m \rangle = \langle 1, \frac{d^n}{dz^n} z^m \rangle = \langle 1, 0 \rangle = 0$$

and the $n < m$ terms vanish by symmetry

$$\langle z^n, z^m \rangle = \overline{\langle z^m, z^n \rangle} = 0.$$

The terms on the diagonal are just as easy

$$\langle z^n, z^n \rangle = \langle 1, \frac{d^n}{dz^n} z^n \rangle = n! \langle 1, 1 \rangle = n!,$$

so, in summary,

$$\langle z^n, z^m \rangle = \begin{cases} n! & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

gives the unique inner product on $\mathbb{C}[z]$ making m_z and $\frac{d}{dz}$ into adjoints.