GRT Seminar Lecture 21 John Baez

Quantum Harmonic Oscillator

$$\begin{tikzcd}
\text{Quantum Harmonic Oscillator} \; \mathbb{C}[z_1, \ldots, z_n] \arrow[rr, "q\text{-deform}"] \arrow[d, shift right, "\text{groupoidify}"'] & & \text{q-Deformed Harmonic Oscillator} \; \mathbb{C}_q[z_1, \ldots, z_n] \arrow[d, "\text{groupoidify}"'] \\
\mathrm{FinSet}^n_0 \simeq [n\text{-colored finite sets}]_0 \arrow[u, shift right, "\text{degroupoidify}"'] \arrow[rr] & & ??
\end{tikzcd}$$

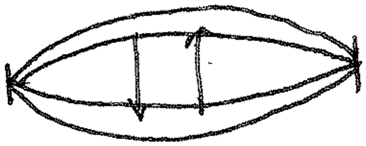
$\simeq$ [finite sets S w. n-stage flag $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S]_0$

where I <u>think</u>

$$?? = \left[\text{finite-dim vector spaces } V \text{ over } \mathbb{F}_q \text{ w. } n\text{-stage flag } \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V\right]_0$$

We'll q-deform & groupoidify the harmonic oscillator using this.

Light in a box



vibrational modes of light — but they're quantized!

Another Harmonic oscillator system is the pendulum

in nice units:

$$\underset{\substack{\text{energy}\\\text{or}\\\text{Hamiltonian}}}{H} = \frac{1}{2}\left(\underset{\text{momentum}}{p^2} + \underset{\text{position}}{q^2}\right)$$

In quantum mechanics, q & p become <u>operators</u> on a Hilbert space $L^2(\mathbb{R})$

$\psi : \mathbb{R} \to \mathbb{C}$

$\|\psi\| = 1$

$\int_U |\psi|^2 dx$ = prob. of finding the position to lie in $U \subseteq \mathbb{R}$

position operator

$$(q\psi)(x) = x\,\psi(x)$$

momentum operator

$$(p\psi)(x) = \frac{1}{i}\frac{d\psi}{dx}$$

As operators, p & q don't commute; instead

$$pq - qp = \frac{1}{i}$$

Now the Hamiltonian becomes an operator

$$H = \frac{1}{2}(p^2 + q^2) : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

with eigenvectors w. eigenvalues $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Homework:

① $\psi_0(x) = e^{-x^2/2}$ has $H\psi_0 = \frac{1}{2}\psi_0$

② Define the creation & annihilation operators by:

$a^* = \frac{1}{\sqrt{2}}(p + iq)$ - creation

$a = \frac{1}{\sqrt{2}}(p - iq)$ - annihilation

Show:

$[a, a^*] := aa^* - a^*a = 1$ & $H = a^*a + \frac{1}{2}$

$[H, a^*] = a^*$ (use: $[p,q] = \frac{1}{i}$)

$[H, a] = -a$

Then use these to show:

$$H\psi = \lambda\psi \implies Ha^*\psi = (\lambda+1)a^*\psi$$
$$\& \ Ha\psi = (\lambda-1)a\psi$$