

Last quarter we introduced the concepts of groupoidification & degroupoidification, and we gave an example having to do with Hecke operators. Today we will consider another example. Degroupoidification is the process

$$\text{Groupoids} \longrightarrow \text{VSP}$$

$$\text{Groupoid - Spans} \longrightarrow \text{Linear Operators}$$

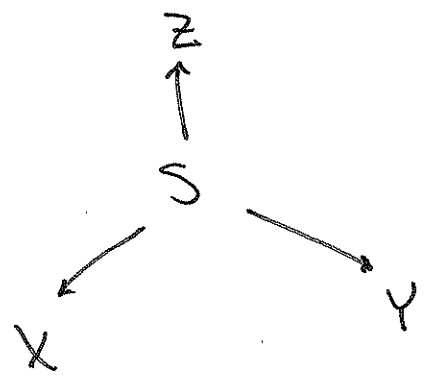
We are going to consider as a Groupoid, an abelian category A , or really, its underlying groupoid A_0 . An example of an abelian category is the category of R -modules, R a ring. The main example we will want to apply our construction to is:

$A =$ finite-dimensional representations of a finite quiver over a finite field.

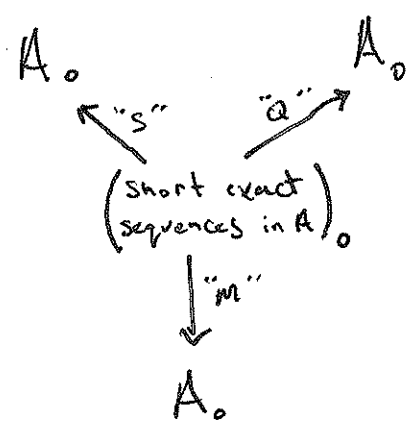
By this process of degroupoidification, we will be trying to construct bilinear operators

$$V \otimes V \longrightarrow V$$

The type of groupoid-span these will come from are tri-spans.



In our example we will have



Let's look at a short exact sequence:

$$0 \longrightarrow S \longrightarrow M \longrightarrow Q \longrightarrow 0$$

In special cases $M = S \oplus Q$, so short exact sequences are generalizations of direct sums.

Let's see what happens under our process of degroupoidification. ⁽³⁾
We should be getting

$$H_0(A_0) \otimes H_0(A_0)$$



$$H_0(A_0)$$

If $A = \underline{\text{FinDim Vect}}_F$, then what we have is polynomial multiplication since A_0 has classes of vector space dimensions i form a basis for the polynomials. So using degroupoidification we have constructed a commutative algebra. In general, given a nice enough abelian category we will hope to get associative algebras. The associative algebra we will be trying to construct is the "Hall algebra of the abelian category A ".

Let's try to understand this multiplication map more concretely.

(4)

First we want to see where associativity comes from.

$$\boxed{0 \longrightarrow S \longrightarrow M} \longrightarrow Q \longrightarrow 0$$

\parallel
 $S \oplus Q$

$$\begin{array}{ccccccc}
 M_0 & \twoheadrightarrow & M_1 & \twoheadrightarrow & M_2 & \twoheadrightarrow & M_3 \\
 \parallel & & & & & & \parallel \\
 0 & & & & & & M
 \end{array}$$

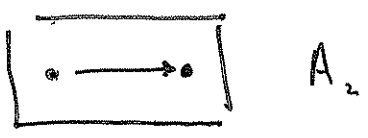
$$M \cong (M_1 + M_2/M_1) + M_3/M_2$$

this sum in the case of finite-dimensional vector spaces is just the arithmetic of dimensions.

This gives us an idea of how to proceed with twisted direct sums of modules. I understand them as "associative".

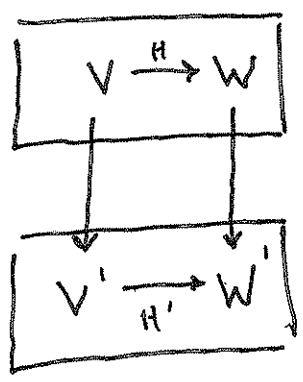
Now to understand this multiplication we will look at quiver representations. A quiver is just a bunch of arrows with some dots along for the ride for the arrows to go to and from.

We are going to represent the following quiver:-



(the 2 stands for 2 dots)

So we consider $Rep_F(A_2)$



a morphism between representations of A_2

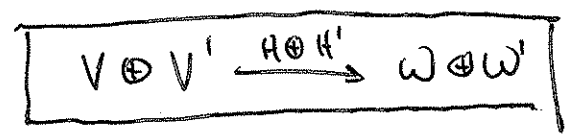
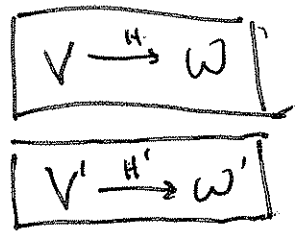
This category turns out to be an abelian category.

So, for our example, our abelian category is

$$A = Rep_F(A_2)$$

i we consider the underlying groupoid A_0 .

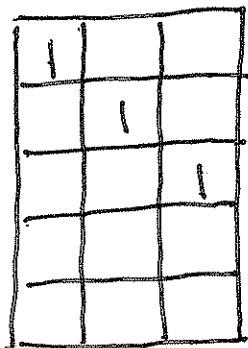
In an abelian category we only need to classify the indecomposable objects.



	V	V'
W	H	0
W'	0	H'

Let's consider

$$\begin{array}{ccc}
 F^3 & \longrightarrow & F^5 \\
 \parallel & & \parallel \\
 F \oplus F \oplus F & & F \oplus F \oplus F \oplus F \oplus F
 \end{array}$$



$$\begin{array}{c}
 \boxed{F \xrightarrow{1} F} \\
 F \xrightarrow{1} F \\
 F \xrightarrow{1} F \\
 \boxed{0 \xrightarrow{0} F} \\
 0 \xrightarrow{0} F
 \end{array}$$

these are indecomposables

the other indecomposable in this abelian category is

$$\boxed{F \xrightarrow{0} 0}$$

So we have classified the objects in this category.