Last quarter we introduced the concepts of groupoidification & degroupoidification, and we gave an example having to do with Hecke operators. Today we will consider another example. Degroupoidification is the process

$$\text{Groupoids} \longrightarrow \text{VSP}$$

$$\text{Groupoid-Spans} \longrightarrow \text{Linear Operators}.$$

We are going to consider as a Groupoid, an abelian category A, or really, its underlying groupoid A_0. An example of an abelian category is the category of R-modules, R a ring. The main example we will want to apply our construction to is:

A = finite-dimensional representations of a finite quiver over a finite field.

By this process of degroupoidification, we will be trying to construct bilinear operators

$$V \otimes V \longrightarrow V.$$

The type of groupoid-span these will come from are tri-spans.

$$\begin{tikzcd} & Z & \\ & S \arrow[u] \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

In our example we will have

$$\begin{tikzcd} A_0 & & A_0 \\ & \begin{pmatrix}\text{short exact} \\ \text{sequences in } A\end{pmatrix}_0 \arrow[ul, "\text{“}S\text{”}"] \arrow[ur, "\text{“}Q\text{”}"'] \arrow[d, "\text{“}M\text{”}"] & \\ & A_0 & \end{tikzcd}$$

Let's look at a short exact sequence:

$$0 \longrightarrow S \longrightarrow M \longrightarrow Q \longrightarrow 0$$

In special cases $M = S \oplus Q$, so short exact sequences are generalizations of direct sums.

Let's see what happens under our process of degroupoidification. We should be getting

$$\begin{array}{c} H_0(A_0) \otimes H_0(A_0) \\ \Big\downarrow \\ H_0(A_0) \end{array}$$

If $A = \underline{\text{FinDimVect}}_F$, then what we have is polynomial multiplication since A_0 has classes of vector space dimensions i form a basis for the polynomials. So using degroupoidification we have constructed a commutative algebra. In general, given a nice enough abelian category we will hope to get associative algebras. The associative algebra we will be trying to construct is the "Hall algebra of the abelian category A".

Let's try to understand this multiplication map more concretely.

First we want to see where associativity comes from.

$$\begin{tikzcd} \boxed{0 \arrow[r] & S \arrow[r] & M} \arrow[d, equal] \arrow[r] & Q \arrow[r] & 0 \\ & & S \oplus Q & & \end{tikzcd}$$

$$\begin{tikzcd} M_0 \arrow[r, tail] \arrow[d, equal] & M_1 \arrow[r, tail] & M_2 \arrow[r, tail] & M_3 \arrow[d, equal] \\ 0 & & & M \end{tikzcd}$$

$$M \approx \left(M_1 + M_2/M_1\right) + M_3/M_2$$

this sum in the case of finite-dimensional vector spaces is just the arithmetic of dimensions. This gives us an idea of how to proceed with twisted direct sums of modules. & understand them as "associative".

Now to understand this multiplication we will look at quiver representations. A quiver is just a bunch of arrows with some dots along for the ride for the arrows to go to and from.

We are going to represent the following quiver :-

$$\boxed{\bullet \longrightarrow \bullet} \quad A_2$$

(the 2 stands for 2 dots)

So we consider $\mathrm{Rep}_F(A_2)$

$$\begin{tikzcd} V \arrow[r, "H"] \arrow[d] & W \arrow[d] \\ V' \arrow[r, "H'"'] & W' \end{tikzcd}$$

a morphism between representations of A_2

This category turns out to be an abelian category.

So, for our example, our abelian category is

$$\mathcal{A} = \mathrm{Rep}_F(A_2)$$

& we consider the underlying groupoid A_0.

In an abelian category we only need to classify the indecomposable objects.

$$\boxed{V \xrightarrow{H} W}$$

$$\boxed{V' \xrightarrow{H'} W'}$$

$$\boxed{V \oplus V' \xrightarrow{H \oplus H'} W \oplus W'}$$

$$\begin{array}{c|c|c|} & V & V' \\ \hline W & H & 0 \\ \hline W' & 0 & H' \\ \hline \end{array}$$

Let's consider

$$\begin{array}{ccc} F^3 & \longrightarrow & F^5 \\ \| & & \| \\ F \oplus F \oplus F & & F \oplus F \oplus F \oplus F \oplus F \end{array}$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & \\ & & \end{bmatrix}$$

$$\boxed{F \xrightarrow{1} F}$$
$$F \xrightarrow{1} F$$
$$F \xrightarrow{1} F$$
$$\boxed{0 \xrightarrow{0} F}$$
$$0 \xrightarrow{0} F$$

these are indecomposables

the other indecomposable in this abelian category is

$$\boxed{F \xrightarrow{0} 0}$$

So we have classified the objects in this category.