Hall Algebra of an Abelian Category

via de-groupoidification

Last time we considered an abelian category that gives an interesting Hall algebra:

$$\mathrm{Rep}_F(A_2) \quad , \quad A_2 = \text{quiver } \bullet \longrightarrow \bullet$$

The Hall algebra is, in some sense, all about treating the "twisted sum" (of objects in an abelian category) as an "operation". This is a little funny because the twisted sum is not an operation. It is not deterministic. You can't form the twisted sum of two objects without some extra information. This information is recorded in the data of a short exact sequence

$$0 \longrightarrow V \longrightarrow X \longrightarrow W \longrightarrow 0 .$$

We consider "spans"

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

which are a generalization of the concept of maps

$$X \longrightarrow Y.$$

This amounts to a non-determinism in a span, where to get from X to Y, we need to go "against the grain". This non-determinism is somehow related to the non-determinism in quantum mechanics with regards to the creation & annihilation operators.

"Hiring is easy, but firing is from the heart."

So creation is in some way non-deterministic, while annihilation involves a choice.

Last time we worked out the indecomposable (or atomic with respect to direct sum) representations in $\mathrm{Rep}_F(A_2)$. All the other representations will be built from these.

$$\underset{C}{F \xrightarrow{1} F} \qquad \underset{B}{F \longrightarrow 0} \qquad \underset{A}{0 \longrightarrow F}$$

We get representations such as

$$7A \oplus 3B \oplus 4C$$

or

$$A \oplus B = F \xrightarrow{0} F$$

We can get an interesting twisted sum of quiver representations:

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F \arrow[r, "1"] \arrow[d, "1"] & F \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & F \arrow[r, "1"'] & F \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So C is similar to the direct sum of A & B, since each are arrows from F to F, but they are different. C is the twisted direct sum of A & B.

Also, C is reducible since it can be written as a twisted direct sum, but A & B are each both indecomposable & irreducible.

Let's remind ourselves how the abstract definition of a Hall algebra works.

$$\text{Hall Alg}(\text{FD Rep}_F(A_2)) =$$

$$H_0(\text{underlying groupoid}(\text{FD Rep}_F(A_2)))$$

as a vector space. Here, secretly, $F = \mathbb{Z}_5$.

Our basis elements will be representations such as we considered earlier

$$7A \oplus 3B \oplus 4C.$$

So to specify a basis, we specify three natural numbers. Our basis is then similar to that of polynomials in three variables, but as we have seen from the short exact sequences our multiplication is not going to be commutative.

Let's think of A_2 as a "Dynkin diagram", which gives us a simple Lie algebra.

The positive part of quantized universal enveloping algebra of $A_2 = \mathfrak{sl}(3)$. This turns out to look a lot like our Hall algebra . . .

We will switch from an additive notation to a multiplicative notation & remember that our multiplication will not be commutative. So

$$7A \oplus 3B \oplus 4C \quad \text{becomes} \quad A^7 \cdot B^3 \cdot C^4 .$$

On second thought, we will use $(7, 3, 4)$.

So, let's see how this works.

$$(1, 0, 0) \cdot (0, 1, 0) \stackrel{?}{=}$$

$$? \cdot (1, 1, 0) + ? \cdot (0, 0, 1)$$

We denote $\mathrm{FDRep}_F(A_2) = X$

$$
\begin{tikzcd}
\boxed{A} \quad X & & X \quad \boxed{B} \\
& SES(X) \arrow[ul, "\text{``subobject''}"'] \arrow[ur, "\text{``quotient object''}"] \arrow[d, "\text{``middle'' = ``the twisted sum''}"] & \\
& X &
\end{tikzcd}
$$

So we are considering all short exact sequences with A on the left & B on the right.

We won't say how to find the coefficients of our multiplication map now, but instead say a word about showing the multiplication is associative.

We need to show the following commutes:

$$
\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "1 \otimes m"] \arrow[d, "m \otimes 1"'] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"'] & V
\end{tikzcd}
$$

SES(X) = 2-stage flags in X

For associativity, we consider 3-stage flags in X. & there are 2 main ways to get 2-stage flags from 3-stage flags.

$$\begin{tikzcd} \text{2-stage flags}(X) \\ \text{3-stage flags}(X) \arrow[u, "(12)3"] \arrow[d, "1(23)"'] \\ \text{2-stage flags}(X) \end{tikzcd}$$

Hopefully, we can work this out next time.