Harmonic Oscillator

$$p, q : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

$$q\psi(x) = x\psi(x)$$

$$p\psi(x) = \frac{1}{i}\psi'(x)$$

$$[p,q] = \frac{1}{i} \qquad a^* = \frac{p+iq}{\sqrt{2}}$$

$$[a,a^*] = 1 \qquad a = \frac{p-iq}{\sqrt{2}}$$

$$H = \frac{1}{2}(p^2+q^2)$$

$$= a^*a + \frac{1}{2}$$

$\psi_0 \in L^2(\mathbb{R})$ given by $\psi_0(x) = e^{-x^2/2}$

has

$$a\psi_0 = 0$$

i.e. $(p - iq)\psi_0 = 0$

$$\left(\frac{1}{i}\frac{d}{dx} - ix\right)\psi_0 = 0$$

$$\frac{d}{dx}e^{-x^2/2} = -xe^{-x^2/2}$$

Thus:

$$H\psi_0 = \left(a^*a + \frac{1}{2}\right)\psi_0 = \frac{1}{2}\psi_0$$

or let

$$N = H - \tfrac{1}{2} = a^* a$$

$$\text{i.e.} \quad N \psi_0 = 0$$

Also:

$$[N, a^*] = a^*$$

$$[N, a] = -a$$

$$N\psi = \lambda \psi \implies N a^* \psi = a^* N \psi + a^* \psi$$
$$= (\lambda + 1) a^* \psi$$

$$\text{i.e.} \quad N\psi = \lambda\psi \implies N a \psi = (\lambda - 1) a \psi$$

So if

$$\psi_n = (a^*)^n \psi_0$$

then

$$N \psi_n = n \psi_n$$

so N is called the <u>number operator</u>.

Thm. — ψ_n form an orthogonal basis of $L^2(\mathbb{R})$.
(a Hilbert space basis)

Notice:

$$a^* \psi_n = \psi_{n+1}$$

and

$$a \psi_n = n \psi_{n-1} \qquad (n \geq 0)$$

which we prove by induction:

$$a \psi_0 = 0$$

$$\text{\& } a\psi_n = n\psi_{n-1} \implies a\psi_{n+1} = a a^* \psi_n$$
$$= a^* a \psi_n + \psi_n$$
$$= a^* n \psi_{n-1} + \psi_n$$
$$= (n+1)\psi_n \,.$$

Thus: $N\psi_n = a^* a \psi_n$
$$= n \psi_n \,.$$

So we have an analogy:

$$\mathbb{C}[z] \xrightarrow{\alpha} L^2(\mathbb{R})$$
$$z^n \longmapsto \psi_n$$

$$m_z z^n = z^{n+1} \qquad a^* \psi_n = \psi_{n+1}$$

$$\frac{d}{dz} z^n = n z^{n-1} \qquad a \psi_n = n \psi_{n-1}$$

α will not be an isomorphism, but it is 1-1 & has a dense range.

HW: Find the (unique) inner product on $\mathbb{C}[z]$ s.t. m_z & $\frac{d}{dz}$ are adjoints,

i.e.: $$\langle z z^n, z^m \rangle = \langle z^n, \tfrac{d}{dz} z^m \rangle$$

$\forall\, n, m \geq 1$, and $\|z\| = 1$.

With this inner product on $\mathbb{C}[z]$,

$$\alpha : \mathbb{C}[z] \longrightarrow L^2(\mathbb{R})$$

is 1-1, inner product preserving, & has a dense range. So the Hilbert space completion of $\mathbb{C}[z]$, <u>Fock space</u>, is equipped with a unitary operator to $L^2(\mathbb{R})$.

Henceforth we'll drop the analysis, replace $\mathbb{C}$ by any field k, & work with $k[z]$ with operators

$$a^* = m_z \qquad a = \frac{d}{dz} \qquad N = a^* a$$

We can easily generalize to $K[z_1, \ldots, z_n]$ with operators

$$a_i^* = m_{z_i} \qquad N_i = a_i^* a_i$$

$$a_i = \frac{\partial}{\partial z_i} \qquad N = \sum_i N_i$$

Now let's groupoidify $K[z]$, a, a^*.

<u>Degroupoidification</u> turns groupoids into vector spaces, spans of groupoids into linear operators.

If X is a groupoid, there's a set $\underline{X}$ of isomorphism classes of objects. We then form a vector space with $\underline{X}$ as its basis — this vector space is called $H_0(X)$, the "zeroth homology".

If we have a functor

$$f : X \longrightarrow Y$$

between groupoids, we get an operator

$$f_* : H_0(X) \longrightarrow H_0(Y)$$
$$[x] \longmapsto [f(x)]$$

and sometimes:

$$f^! : H_0(Y) \longrightarrow H_0(X)$$
$$[y] \longmapsto |\mathrm{Aut}(y)| \sum_{[x] \in \underline{f^{-1}(y)}} \frac{[x]}{|\mathrm{Aut}(x)|}$$

Here $f^{-1}(y)$ is the groupoid whose objects are $x \in X$ with $f(x) \cong y$ & whose morphisms are all morphisms between those. $\mathrm{Aut}(x)$ is the set of all isomorphisms from x to x. $\underline{f^{-1}(x)}$ is the set of isomorphism classes in $f^{-1}(x)$.

Example: FinSet_0 has $H_0(\mathrm{FinSet}_0) = k[z]$ & consider

$$\mathrm{FinSet}_0 \xrightarrow{+1} \mathrm{FinSet}_0$$

$$\begin{array}{ccc} S & & S \sqcup \{*\} \\ \alpha \downarrow & \longmapsto & \downarrow \alpha+1 \\ T & & T \sqcup \{*\} \end{array}$$

What's

$$H_0(\mathrm{Fin\,Set}_0) \xrightarrow{(+1)_*} H_0(\mathrm{Fin\,Set}_0)$$

$$z^n \longmapsto z^{n+1}$$

It's the creation operator. What's

$$H_0(\mathrm{FinSet}_0) \xrightarrow{(+1)^!} H_0(\mathrm{FinSet}_0)$$

$$z^n \longmapsto ??$$

Say y is an n-element set, so $[y] = z^n$.

$$[x] = \{(n-1)\text{-element sets}\} = z^{n-1}$$

So, $|\mathrm{Aut}(x)| = (n-1)!$ & $|\mathrm{Aut}(y)| = n!$

$$(+1)^! \, z^n = n! \, \frac{z^{n-1}}{(n-1)!} = n \, z^{n-1}$$

It's the annihilation operator!