Geometric Representation Theory Jim Dolan 1/24/08 Lecture 25

Hall Algebra

"Twisted sum"

JB has been talking about the quantum harmonic oscillator. The annihilation operator & the twisted sum are each nondeterministic in some ways.

Hall Algebra

- degroupoidification of the abelian category of reps of some quiver over a finite field K, $Rep_K(Q)$
- Part of a quantum group

This is saying that this corresponding part of a quantum group can be degroupoidified.

We are going to do some calculations working with the quiver $Q = A_2$

$$\bullet \longrightarrow \bullet$$

We found three indecomposables, one of which is a twisted sum: $A + B = C$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & K & \xrightarrow{1} & K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & A & & C & & B & &
\end{array}$$

$$\left[\begin{smallmatrix} 0 \\ \downarrow \\ K \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} K \\ \downarrow \\ 0 \end{smallmatrix}\right] = ? \left[\begin{smallmatrix} K \\ 0\downarrow \\ K \end{smallmatrix}\right] + ? \left[\begin{smallmatrix} K \\ 1\downarrow \\ K \end{smallmatrix}\right]$$

$$\text{Hall Alg}(\text{Rep}_K(Q)) = H_0(\text{Rep}_K(Q))$$

$$G \xrightarrow{F} H \qquad\qquad H_0(G) \xleftarrow{F^!} H_0(H)$$

Y component of H

$$F^!(Y) = \frac{\displaystyle\sum_{\substack{X \text{ component} \\ \text{of } F^{-1}(Y)}} |X| \cdot X}{|Y|}$$

Notation: $\underline{\mathbb{A}} = \mathrm{Rep}_k(Q)$

We have a span

$$\begin{array}{ccc} & G = SES(\underline{\mathbb{A}}) & \\ \text{"Twisted summands" } (S,Q) \; \swarrow F & & M \searrow \text{ "Twisted sum"} \\ H = \underline{\mathbb{A}} \times \underline{\mathbb{A}} & & \underline{\mathbb{A}} \end{array}$$

$$0 \longrightarrow S \longrightarrow M \longrightarrow Q \longrightarrow 0$$

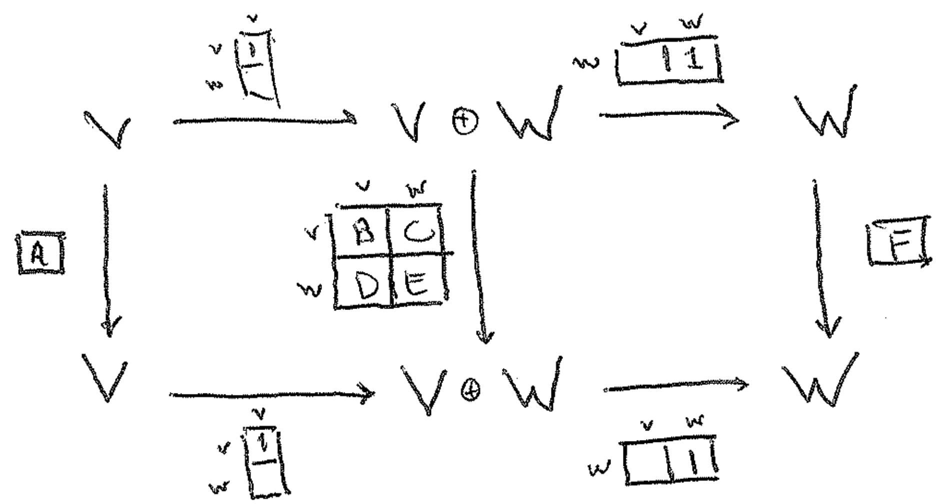
$$\begin{array}{ccc} H_0(H) & \xrightarrow{F^!} & H_0(G) \\ \| & & \| \\ H_0(\underline{\mathbb{A}}) \otimes H_0(\underline{\mathbb{A}}) & & H_0(SES(\underline{\mathbb{A}})) \end{array}$$

$$\begin{bmatrix} 0 \\ \downarrow \\ k \end{bmatrix} \otimes \begin{bmatrix} k \\ \downarrow \\ 0 \end{bmatrix} \longmapsto \; ? \begin{bmatrix} 0 & \to & k & \xrightarrow{1} & k \\ \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow \\ k & \xrightarrow[1]{} & k & \to & 0 \end{bmatrix} + \; ? \begin{bmatrix} 0 & \to & k & \to & k \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\ k & \to & k & \to & 0 \end{bmatrix}$$

(untwisted) (twisted)

Let's try to figure out this first coefficient.

Let's first see how to calculate for a split short exact sequence.



$$A = B \qquad D = 0$$

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} D & E \end{bmatrix} = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} F \end{bmatrix} = \begin{bmatrix} 0 & F \end{bmatrix}$$

$$\begin{bmatrix} A & C \\ 0 & F \end{bmatrix} \qquad |\mathrm{Aut}(V)| \cdot |\mathrm{Aut}(W)| \cdot |\mathrm{Hom}(W, V)|$$

Now back to our calculation.

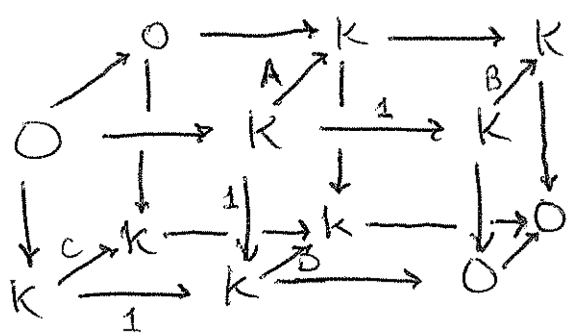
$$\frac{\left|\left[\begin{array}{ccccc} 0 & \longrightarrow & k & \xrightarrow{1} & k \\ \downarrow & & {\scriptstyle 0}\downarrow & & \downarrow \\ k & \xrightarrow[1]{} & k & \longrightarrow & 0 \end{array}\right]\right|}{\left|\left[\begin{array}{c} 0 \\ \downarrow \\ k \end{array}, \begin{array}{c} k \\ \downarrow \\ 0 \end{array}\right]\right|} = \frac{\left|\operatorname{Aut}\left(\begin{array}{c} 0 \\ \downarrow \\ k \end{array}, \begin{array}{c} k \\ \downarrow \\ 0 \end{array}\right)\right|}{\left|\operatorname{Aut}\left(\begin{array}{ccccc} 0 & \longrightarrow & k & \xrightarrow{1} & k \\ \downarrow & & {\scriptstyle 0}\downarrow & & \downarrow \\ k & \xrightarrow[1]{} & k & \longrightarrow & 0 \end{array}\right)\right|}$$

$$= \frac{\left|\operatorname{Aut}\left(\begin{array}{c} 0 \\ \downarrow \\ k \end{array}\right)\right| \cdot \left|\operatorname{Aut}\left(\begin{array}{c} k \\ \downarrow \\ 0 \end{array}\right)\right|}{\left|\operatorname{Aut}\left(\begin{array}{c} 0 \\ \downarrow \\ k \end{array}\right)\right| \cdot \left|\operatorname{Aut}\left(\begin{array}{c} k \\ \downarrow \\ 0 \end{array}\right)\right| \cdot \left|\operatorname{Hom}\left(\begin{array}{c} k \\ \downarrow \\ 0 \end{array}, \begin{array}{c} 0 \\ \downarrow \\ k \end{array}\right)\right|} = 1$$

This is our coefficient for

$$\left[\begin{array}{ccccc} 0 & \longrightarrow & k & \xrightarrow{1} & k \\ \downarrow & & {\scriptstyle 0}\downarrow & & \downarrow \\ k & \xrightarrow[1]{} & k & \longrightarrow & 0 \end{array}\right].$$

Now, the second coefficient:



$A = B = C = D$

$$\frac{\left|\mathrm{Aut}\begin{pmatrix} 0 \\ \downarrow \\ k \end{pmatrix}\right| \cdot \left|\mathrm{Aut}\begin{pmatrix} k \\ \downarrow \\ 0 \end{pmatrix}\right|}{\left|\mathrm{Aut}_k(k)\right|} = \frac{(Q-1)^2}{(Q-1)}$$

So, our second coefficient is $Q-1$.

$$(Q-1)\left[\begin{array}{ccccc} 0 & \longrightarrow & k & \longrightarrow & k \\ \downarrow & & \downarrow 1 & & \downarrow \\ k & \longrightarrow & k & \longrightarrow & 0 \end{array}\right]$$

$\mathrm{Ext}^1(W, V) =$ "set of isomorphism classes of extensions of V by W"

$$W = \begin{array}{c} k \\ \downarrow \\ 0 \end{array}$$

$$V = \begin{array}{c} 0 \\ \downarrow \\ k \end{array}$$

It would be interesting to try to get Hall algebra as all of some quantum group.