Given a span of groupoids

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

we get an operator:

$$\tilde{S} : H_0(X) \longrightarrow H_0(Y)$$

$$\tilde{S} = q_* p^!$$

Given composable spans

$$\begin{tikzcd} & & TS \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[rr, phantom, "\smile"] & & T \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ X & & Y & & Z \end{tikzcd}$$

we compose them via weak pullback

$$TS = [s \in S, t \in T, \alpha : q(s) \longrightarrow p'(t)]$$

Thm. — $\widetilde{TS} = \tilde{T}\tilde{S}$

See paper by J. Morton for proof.

We have an identity span

$$1_X = \begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "1"] & \\ X & & X \end{tikzcd}$$

Thm. — $\widetilde{1_X} = 1_{H_0(X)}$

Given spans

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} , \quad \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

we can <u>add</u> them:

$$\begin{tikzcd} & S+T \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Given groupoids S & T, they have a coproduct, or disjoint union, $S+T$.

Thm. – $\widetilde{S+T} \simeq \widetilde{S} + \widetilde{T}$

Annihilation & Creation Operators

We have:

$$H_0(\mathrm{FinSet}_0) \cong K[z]$$

(a field of char. zero

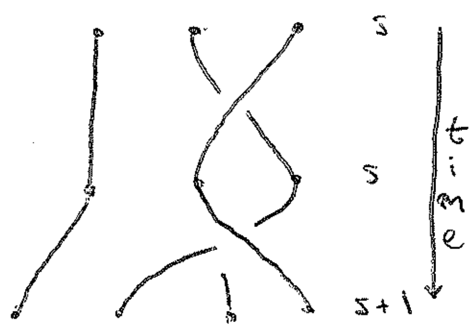
& certain spans:

$$A^* = \begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

$$A = \begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

which we can draw as Feynman diagrams:

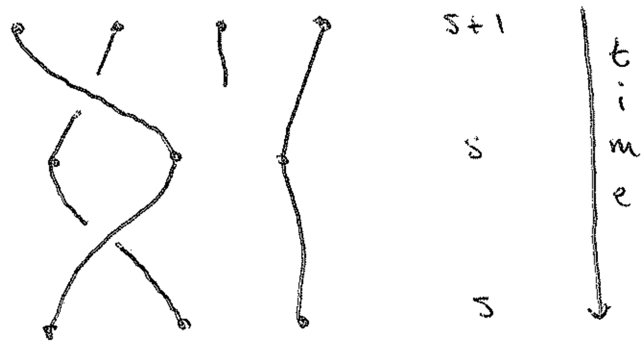
A^* :



$$\begin{tikzcd} & S \in \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & \\ S \in \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \ni S+1 \end{tikzcd}$$

$\longrightarrow$ time

A:



$$\begin{tikzcd}
 & s \in \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & \\
s+1 \in \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \ni s
\end{tikzcd}$$

$$\xrightarrow[\text{time}]{\hspace{4em}}$$

Now let's show:

$$AA^* \simeq A^*A + 1$$

& thus

$$\widetilde{AA^*} = \widetilde{A^*A + 1}$$

$$\widetilde{A}\,\widetilde{A}^* = \widetilde{A}^*\widetilde{A} + 1$$

$$aa^* = a^*a + 1$$

$$\begin{tikzcd}
 & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & Y
\end{tikzcd} = S$$

$$\begin{tikzcd}
 & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Y & & X
\end{tikzcd} = S^*$$

$$H_0(X) \xrightarrow{\tilde{S}} H_0(Y)$$

$$H_0(Y) \xrightarrow{\tilde{S}^*} H_0(X)$$

$$\tilde{S}^* = \widetilde{S^*}$$

A^*A:

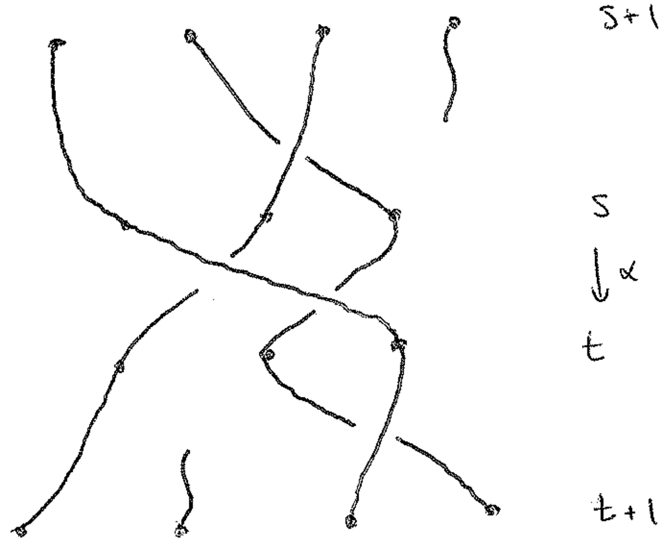
$$\begin{tikzcd}
 & & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
 & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

Via Feynman diagrams:

The groupoid on top is the weak pullback:

$$[s, t \in \mathrm{FinSet}_0,\ \alpha : s \xrightarrow{\sim} t] \simeq \mathrm{FinSet}_0$$

A Feynman diagram for A^*A:



AA^*:

$$\begin{tikzcd}
 & & {[s, t \in \mathrm{FinSet}_0,\ \alpha : s+1 \xrightarrow{\sim} t+1]} \arrow[dl, "{(s,t,\alpha) \mapsto s}"'] \arrow[dr, "{(s,t,\alpha) \mapsto t}"] & & \\
 & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

We want to show

$$[s, t \in \mathrm{FinSet}_0,\ \alpha : s+1 \xrightarrow{\sim} t+1] \simeq [s, t \in \mathrm{FinSet}_0,\ \alpha : s \xrightarrow{\sim} t] + [s, t \in \mathrm{FinSet}_0,\ \alpha : s+1 \xrightarrow{\sim} t+1 \text{ s.t. } \alpha(1) \neq 1]$$

$$\left[s, t \in \mathrm{FinSet}_0,\ \alpha : s+1 \xrightarrow{\sim} t+1 \ \text{ s.t. } \alpha(1) \neq 1\right]$$

$$\wr\!\mid$$

$$\left[s', t' \in \mathrm{FinSet}_0,\ \alpha : s' \xrightarrow{\sim} t'\right] \qquad \begin{aligned} s' &= s - \alpha^{-1}(1) \\ t' &= t - \alpha(1) \end{aligned}$$

$$\wr\!\mid$$

$$\mathrm{FinSet}_0$$

AA^*:

$$\begin{tikzcd} & {[s, t \in \mathrm{FinSet}_0,\ \alpha : s+1 \to t+1]} \arrow[dl, "(s,t,\alpha) \mapsto s"'] \arrow[dr, "(s,t,\alpha) \mapsto t"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

This span is equivalent to a sum:

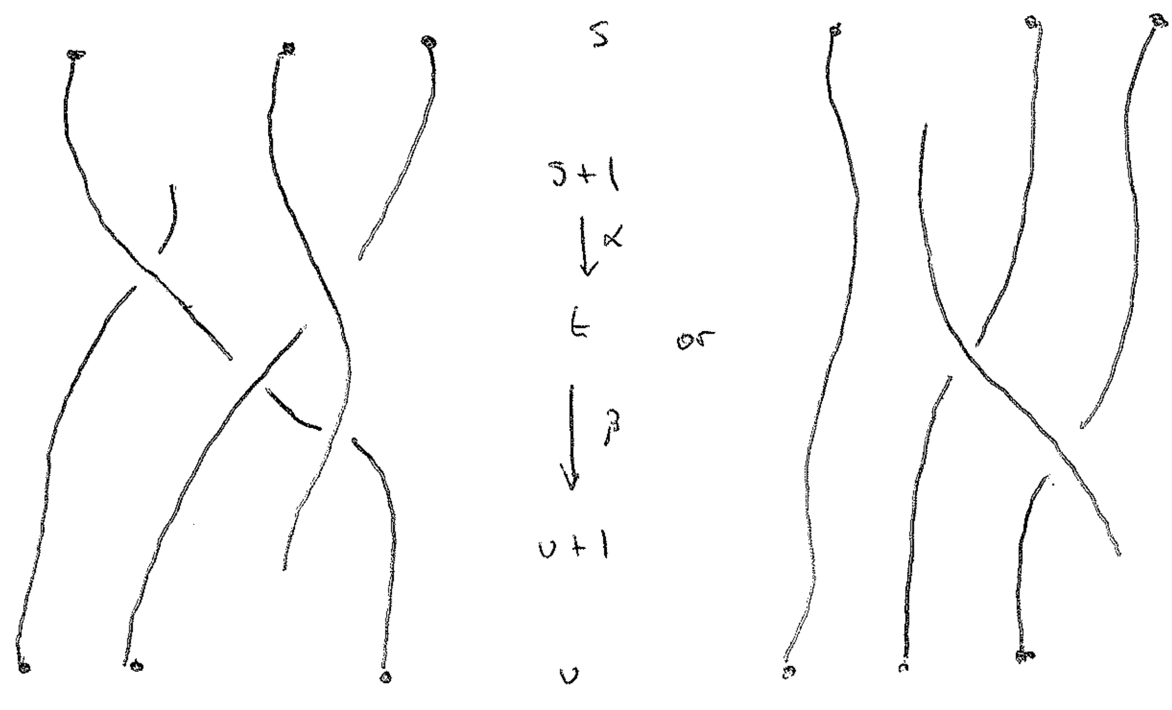
$$\begin{tikzcd} & {[s, t \in \mathrm{FinSet}_0 : \alpha : s \xrightarrow{\sim} t]} \arrow[dl, "(s,t,\alpha) \mapsto \alpha"'] \arrow[dr, "(s,t,\alpha) \mapsto t"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd} + \begin{tikzcd} & {[s', t' \in \mathrm{FinSet}_0 : \alpha : s' \to t']} \arrow[dl, "(s',t',\alpha) \mapsto s'+1"'] \arrow[dr, "(s',t',\alpha) \mapsto t'+1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

This sum of spans is equivalent to:

$$1_{\mathrm{FinSet}_0} + A^*A .$$

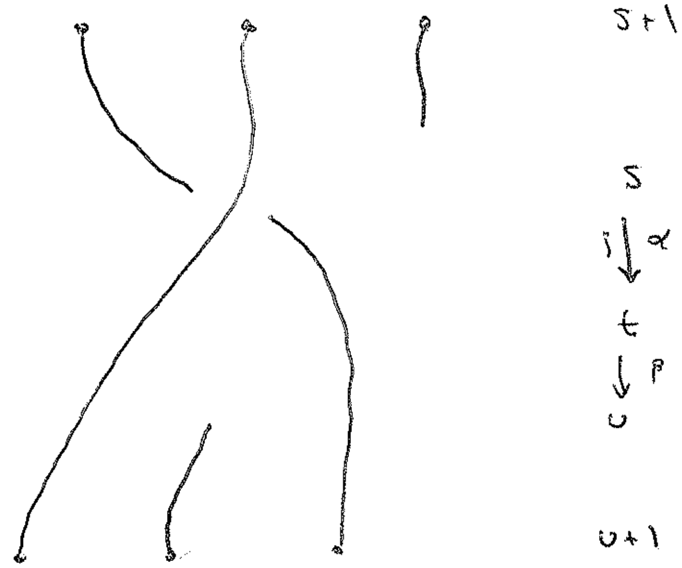
Or via Feynman diagrams:

AA^* comes in two cases:



This is A^*A $\qquad$ This is 1_{FinSet_0}

A^*A:



Making the open strings longer we can change the order of annihilation & creation.

We are going to generalize all of this from FinSet_0 to FinSet_0^n & use this to groupoidify the action of $\mathfrak{gl}(n)$ on $k[z_1, \ldots, z_n]$.