Hall Algebras

We are going to think about modules of Hall algebras, Hall modules, & since Hall algebras are secretly part of Quantum groups, Hall modules will secretly be representations of Quantum groups.

Associativity of Hall algebras:

$$V \otimes V \xrightarrow{\circ} V$$

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\circ \otimes V"] \arrow[d, "V \otimes \circ"'] & V \otimes V \arrow[d, "\circ"] \\
V \otimes V \arrow[r, "\circ"'] & V
\end{tikzcd}$$

We have to remember V comes from a groupoid & $\circ$ comes from a trispan of groupoids.

$$V = H_0(\underbrace{\mathrm{Rep}_F(Q)}_{\substack{\text{Abelian} \\ \text{Category} \\ A}}{}_0) \qquad Q = A_2 = \boxed{\circ \longrightarrow \circ}$$

$$
\begin{tikzcd}
A & & A \\
& \mathrm{SES}(A) \arrow[ul, "``S"'] \arrow[ur, "``Q"] \arrow[d, "``m"] & \\
& A &
\end{tikzcd}
$$

$$\boxed{0 \longrightarrow S \longrightarrow M \to Q \to 0}$$

As a bispan we have:

$$
\begin{tikzcd}
& \mathrm{SES}(A) \arrow[dl, "``(S,Q)"'] \arrow[dr, "``m"] & \\
A \times A & & A
\end{tikzcd}
$$

$$\boxed{H_0(A) = V}$$

$$
\begin{tikzcd}
& & \mathrm{SES}(A) \times A \arrow[dl] \arrow[dr] & & \\
& A \times A \times A & & A \times A & \\
A \times \mathrm{SES}(A) \arrow[ur] \arrow[dr] & & & & \mathrm{SES}(A) \arrow[ul] \arrow[dl] \\
& A \times A & & A & \\
& & \mathrm{SES}(A) \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Short exact sequences are also 2-stage flags:

$$2SF(A) = SES(A)$$

$$S \subseteq M$$

$$S \qquad M/S$$

For associativity we want to consider 3-stage flags (or 3-fold twisted sums)

$$S_1 \subseteq S_2 \subseteq M$$

$$S_1 \qquad S_2/S_1 \qquad M/S_2$$

(*)

$$\begin{tikzcd}
 & & 2SF(A) \times A \arrow[dl] \arrow[dr] & & 3SF(A) \arrow[ll] \arrow[dd] \arrow[ddddllll, equal] \\
 & A \times A \times A & & A \times A & \\
A \times 2SF(A) \arrow[ur] \arrow[dr] & & & & 2SF(A) \arrow[ul] \arrow[dl] \\
 & A \times A & & A & \\
3SF(A) \arrow[uu] \arrow[rr] & & 2SF(A) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Think of this as:

$$\begin{tikzcd}
3SF(A) \arrow[r, equal] \arrow[d] \arrow[dr] & 3SF(A) \arrow[dl] \arrow[d] \\
A \times A \times A & A
\end{tikzcd}$$

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & 2SF(A) \arrow[d] & E \subseteq F \arrow[d, mapsto] \\ 2SF(A) \times A \arrow[r] & A \times A & (E, F/E) \\ (B \subseteq C, D) \arrow[r, mapsto] & (C, D) & \end{tikzcd}$$

So,

$$? = \{ (B \subseteq C) \subseteq F \}$$

We can check that $(*)$ commutes & this is a proof of associativity of the Hall algebra.

Hall module $\otimes$ Hall algebra $\longrightarrow$ Hall module

A_0

$V_1 \longrightarrow V_2$, V_1, V_2 vector spaces over $\mathbb{F}$

W, vector space over $\mathbb{F}$

$B = \boxed{\text{"Rep of quiver } A_2 \text{ monically over } W\text{"}}$

$$\begin{tikzcd}
V_1 \arrow[rr] \arrow[dr, tail] & & V_2 \arrow[dl, tail] \\
& W &
\end{tikzcd}$$

We have a trispan

$$\begin{tikzcd}
B_0 & & A_0 \\
& SES(B, A) \arrow[ul, "\text{"S"}"] \arrow[ur, "\text{"Q"}"'] \arrow[d, "\text{"M"}"] & \\
& B_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & S_1 \arrow[r, tail] \arrow[d] & S_2 \arrow[r, tail] \arrow[d] & W \arrow[d, equal] & \text{(lives in } B_0\text{)} \\
 & M_1 \arrow[r, tail] \arrow[d] & M_2 \arrow[r, tail] \arrow[d] & W \arrow[d] & \text{(lives in } B_0\text{)} \\
 & Q_1 \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & 0 \arrow[d] & \text{(lives in } A_0\text{)} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

S_0:

Hall module $= H_0(B_0)$

$\mathrm{Dim}(W)$	$\mathrm{Dim}(H_0(B_0))$
0	1
1	3
2	6
3	10
4	15
5	21