

Hall Algebras

We are going to think about modules of Hall algebras, Hall modules, & since Hall algebras are secretly part of Quantum groups, Hall modules will secretly be representations of Quantum groups.

Associativity of Hall algebras:

$$V \otimes V \xrightarrow{\cdot} V$$

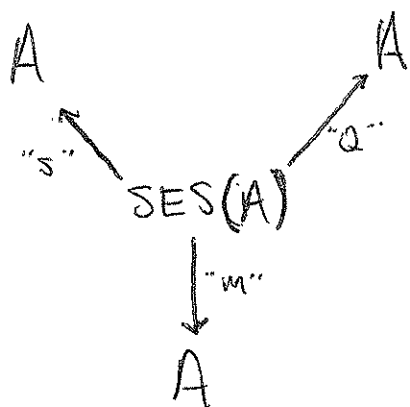
$$V \otimes V \otimes V \xrightarrow{\cdot \otimes V} V \otimes V$$

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{\cdot \otimes V} & V \otimes V \\ \downarrow \cdot & & \downarrow \cdot \\ V \otimes V & \xrightarrow{\cdot} & V \end{array}$$

We have to remember V comes from a groupoid
& \cdot comes from a trispan of groupoids.

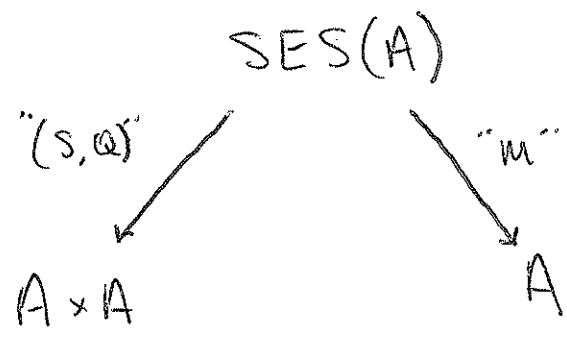
$$V = H_0(\underbrace{\text{Rep}_F(Q)}_{\substack{\text{Abelian} \\ \text{category} \\ A}})$$

$$Q = A_2 = \boxed{\cdot \longrightarrow \cdot}$$

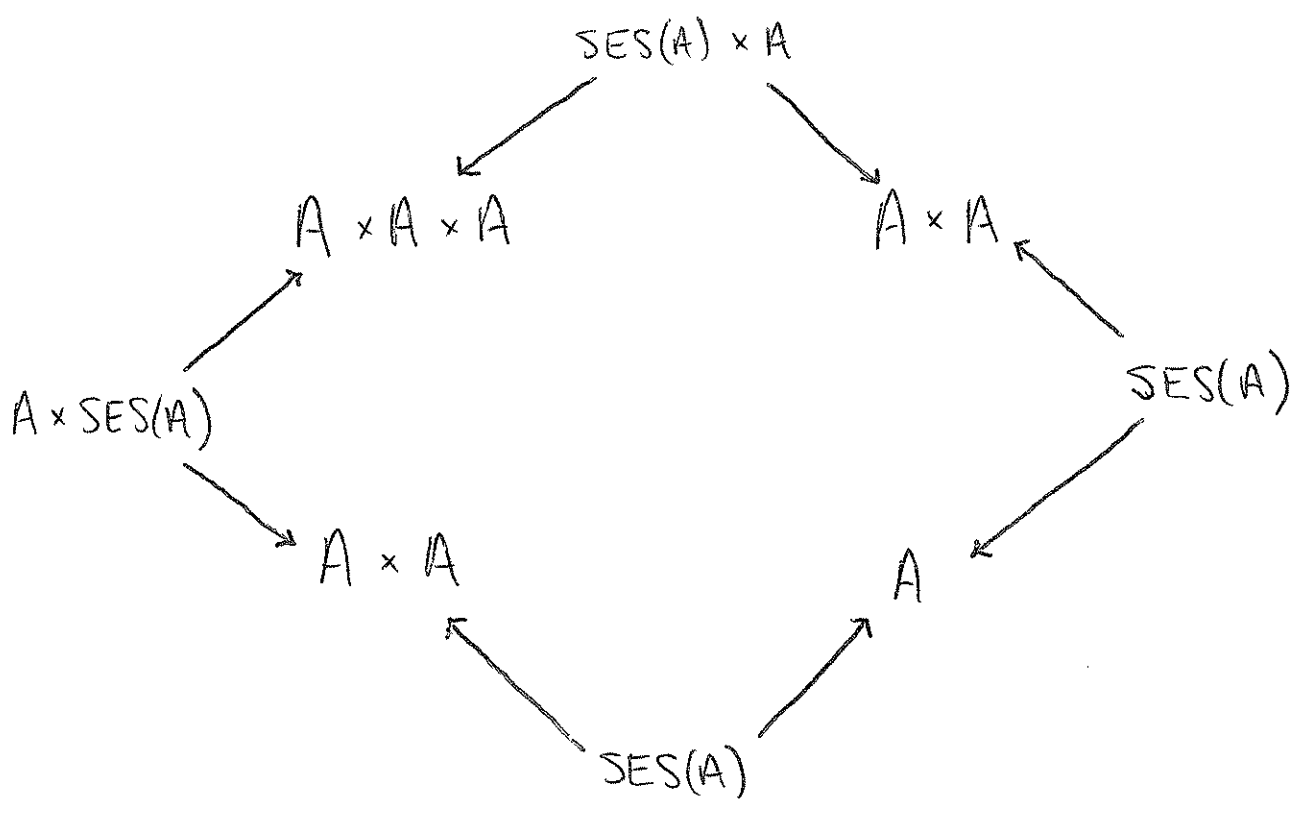


$$0 \rightarrow S \rightarrow M \rightarrow Q \rightarrow 0$$

As a bispan we have:



$$H_0(A) = V$$



Short exact sequences are also 2-stage flags:

$$2SF(A) = SES(A)$$

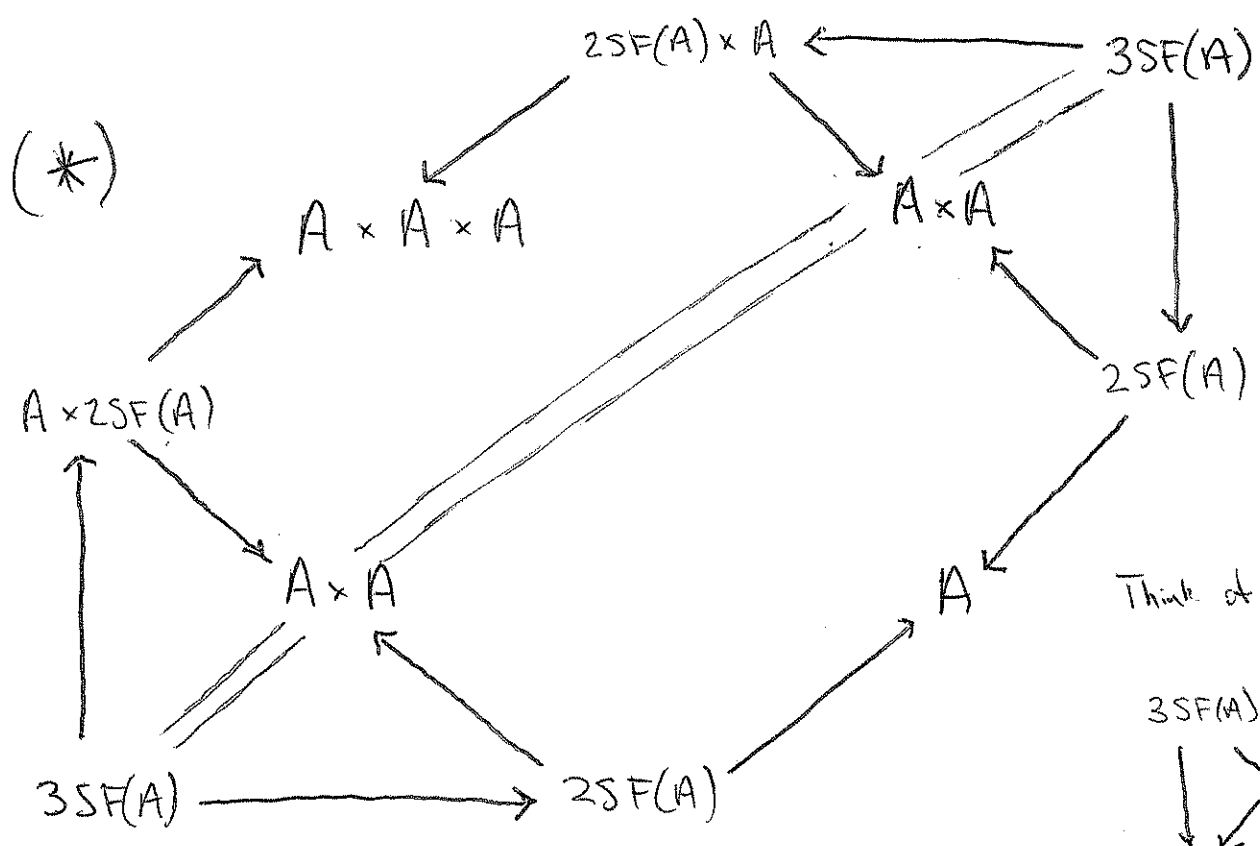
$$S \subseteq M$$

$$S \quad M/S$$

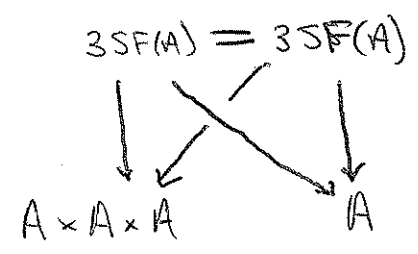
For associativity we want to consider 3-stage flags (or 3-fold twisted sums)

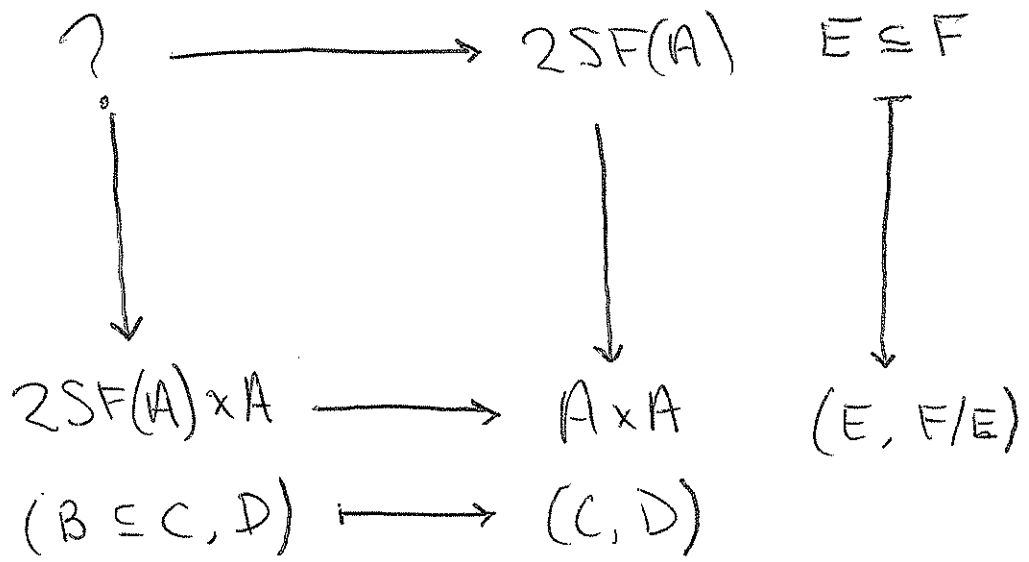
$$S_1 \subseteq S_2 \subseteq M$$

$$S_1 \quad S_2/S_1 \quad M/S_2$$



Think of this as:





So,

$$? = \left\{ (B \subseteq C) \in F \right\}$$

We can check that $(*)$ commutes \therefore this is a proof of associativity of the Hall algebra.

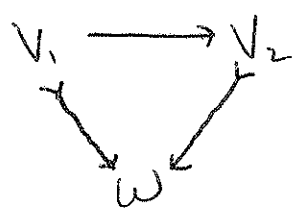
Hall module \otimes Hall algebra \longrightarrow Hall module

A_0

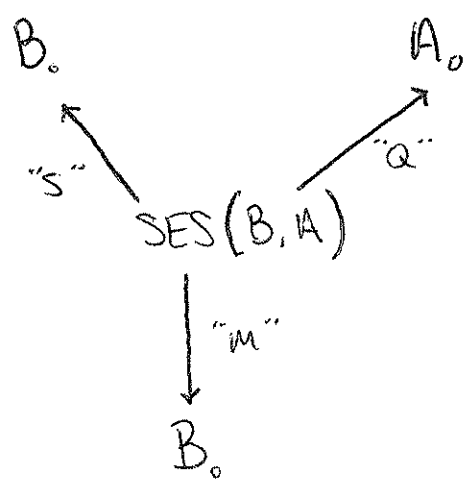
$V_1 \longrightarrow V_2$, V_1, V_2 vector spaces over F

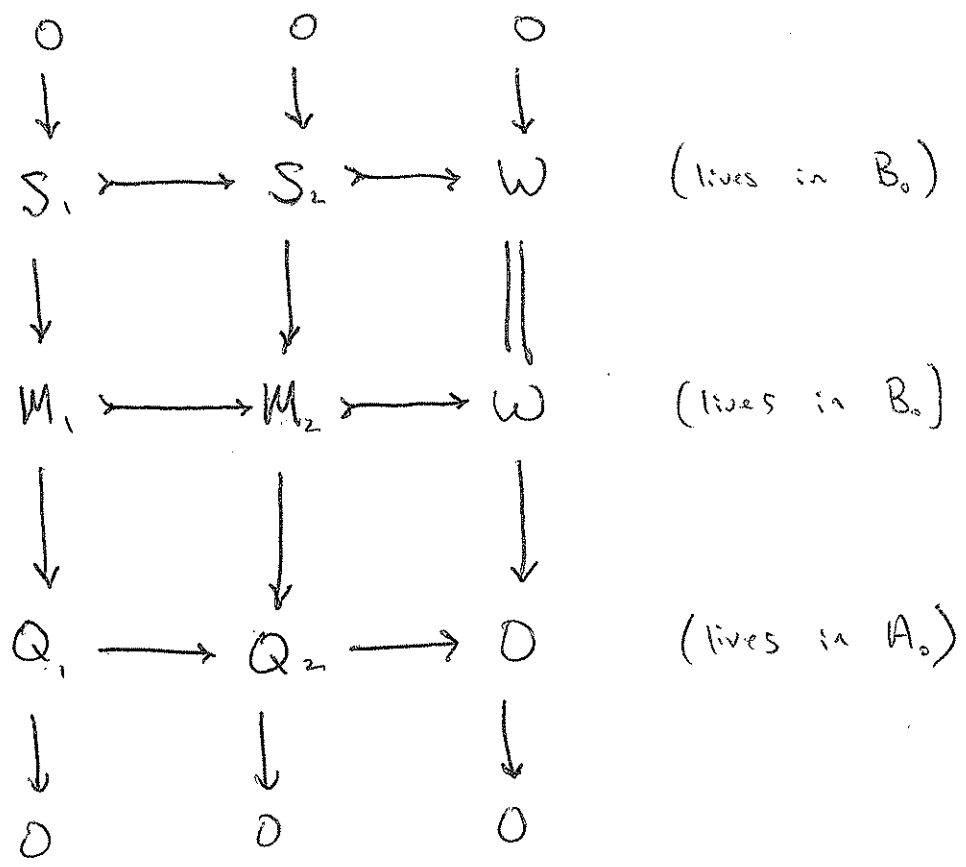
W , vector space over F

$B = \boxed{\text{"Rep of quiver } A_2 \text{ monically over } W"}$



We have a trispan





$S_0:$

Hall module = $H_0(B_0)$

$\text{Dim}(W)$	$\text{Dim}(H_0(B_0))$
0	1
1	3
2	6
3	10
4	15
5	21