Geometric Representation Theory seminar, Lecture 31

February 12, 2008

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1 E_8

Today Bertam Kostant is giving a talk on the Lie group E_8, the largest of the exceptional Lie groups. Imagine trying to pack pennies in 2-dimensions in the densest possible way. Mathematically, the centers of the pennies form a lattice L, which is a subgroup of $\mathbb{R}^2$, with hexagonal symmetry. This is called the A_2 lattice. People keep track of various lattices with Dynkin diagrams:

$$\cdot \longrule \cdot$$

The dots are basis vectors which we can think of as unit vectors. The edge means there is a 120 degree angle between them. We call this a simply laced Dynkin diagram. In three dimensions, this is the problem of the citrus industry. Packing oranges leads to a lattice

$$L \subseteq \mathbb{R}^3$$

This is called the A_3 lattice:

$$\cdot_{e_1} \longrule \cdot_{e_2} \longrule \cdot_{e_3}$$

The angle between e_i and e_{i+1} is 120 deegrees, but between e_1 and e_3 it is 90 degrees, which is encoded by the lack of an edge connecting the two.

$$\cdot \longrule \cdot \qquad\qquad SU(3) \subseteq SL(3, \mathbb{C})$$

Maximal torus in $SU(3)$:

$$T^2 \cong \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \big| |\alpha|, |\beta|, |\gamma| = 1, \alpha\beta\gamma = 1 \right\} \subseteq SU(3)$$

$$\begin{tikzcd} & \mathfrak{su}(3) \arrow[r, "exp"] \arrow[d, "\subseteq"'] & SU(3) \arrow[d, "\subseteq"] \\ L \arrow[r, "\iota"] & \mathfrak{h} \arrow[r, "exp"'] & T^2 \end{tikzcd}$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} \big| a, b, c \in \mathbb{R}, a + b + c = 0 \right\}$$

gets sent by exponentiation to

$$\begin{pmatrix} e^{ia} & 1 & 1 \\ 1 & e^{ib} & 1 \\ 1 & 1 & e^{ic} \end{pmatrix}$$

The kernel of $exp\colon \mathfrak{h} \to T^2$ is

$$L = \left\{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} \big| a, b, c \in 2\pi\mathbb{Z}, a + b + c = 0 \right\}$$

Draw $(a, b, c) \in \mathbb{Z}$ with $a + b + c = 0$

$\cdot(1,-1,0)$

$\cdot(1,0,-1)$ $\cdot(0,-1,1)$

$\cdot(0,0,0)$

$\cdot(0,1,-1)$ $\cdot(-1,0,1)$

$\cdot(-1,1,0)$

This is the A_2 lattice.

$SU(n)$ has a maximal torus:

$$T^{n-1} = \left\{ \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} | \alpha_i \in \mathbb{C}, |\alpha_i| = 1, \alpha_1 \cdots \alpha_n \right\}$$

with Lie algebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} ia_1 & & 0 \\ & \ddots & \\ 0 & & ia_n \end{pmatrix} | a_i \in \mathbb{R}, a_1 + \cdots + a_n = 0 \right\}$$

and kernel of $exp \colon \mathfrak{t} \to T^{n-1}$ is

$$L = \left\{ \begin{pmatrix} 2\pi i a_1 & & 0 \\ & \ddots & \\ 0 & & 2\pi i a_n \end{pmatrix} | a_i \in \mathbb{Z}, a_1 + \cdots + a_n = 0 \right\}$$

with $L \subseteq \mathfrak{t}^{n-1} \cong \mathbb{R}^{n-1}$. This is the A_{n-1} lattice.

$\cdot$ —— $\cdot$ —— $\cdot$ —— $\cdot$ —— $\cdot$

$n = 5$. Check this!

In $4d$, the densest known packing is the D_4 lattice - not the A_4!

$$D_4 = \left\{ (a, b, c, d) \in \mathbb{R}^4 | either a, b, c, d \in \mathbb{Z} or a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}$$

Which vectors in D_4 are closest to the origin?

$$\begin{array}{ll} (\pm 1, 0, 0, 0) & \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \\ (0, \pm 1, 0, 0) & \\ (0, 0 \pm 1, 0) & \\ (0, 0, 0, \pm 1) & \end{array}$$

There are 24 total - vertices of the 24-cell: a hypercube combined with a hypercube.

It turns out that there is another way to think about the D_4 lattice which allows you to generalize to arbitrary dimensions. In n dimensions we can define the D_n lattice using a different coordinate system as a "checkerboard lattice" -

$$D_n = \left\{(a_1, \ldots, a_n) \in \mathbb{Z}^n \text{ such that } \sum a_i \in 2\mathbb{Z}\right\}$$

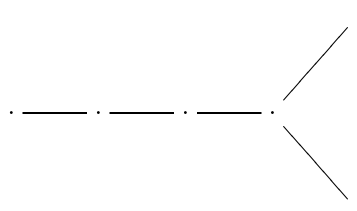
$$D_2 \cong \mathbb{Z}^2$$

$$D_3 \cong A_3$$

$$D_4 \cong \text{old } D_4 \ncong A_4$$

as lattices in an inner product space.
In 5 dimensions, D_5 is the densest.

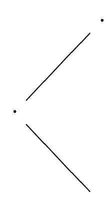
D_n :



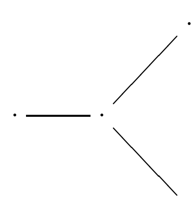
$(n = 6)$

$$D_2 \cong A_1 \oplus A_1$$

$$D_3 \cong A_3$$

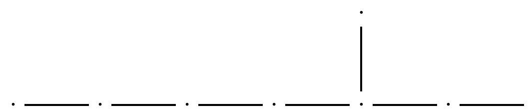


$$D_4$$



This corresponds to rotations in $\mathbb{R}^8$!
D_n corresponds to the Lie group $\mathrm{SO}(2n)$. In 8 dimensions, you could try D_8, but the gaps between spheres are big enough to slip in more spheres of the same size! This gives E_8.

$$E_8 = \left\{(a_1, \ldots, a_n) | a_1 + \cdots + a_8 \in 2\mathbb{Z} \text{ and either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2}\right\}$$



Which vectors in E_8 are closest to the origin:

$$(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0) \quad \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

There are $\begin{pmatrix} 8 \\ 2 \end{pmatrix} \cdot 2^2$ of the first type, i.e. $28 \times 4 = 112$. There are $2^7 = 128$ of these. There are $112 + 128 = 240$ vectors nearest to the origin.

In all these cases, dim(Lie group) = number of dots in Dynkin diagram plus number of vectors closest to the origin = dim(maximal torus).

A_2

$$dim(SU(3)) = 2 + 6 = 8$$

$$dim(\text{the group corr. to } E_8) = 8 + 240 = 248$$