Geometric Representation Theory seminar, Lecture 32

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1

$SL(3) = A_2$

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We want to know the dimensions of the representations obtained by combining Young diagram boxes horizontally. They correspond to the pyramid numbers. Vertical composition is just the tensor power of the rows. The dimension comes from the product of the dimensions of the rows.
We recall the diagram from lecture 30:

$$\begin{tikzcd}
 & P \arrow[dl] \arrow[dr] & \\
{[\underline{Mon}_{Const}(A)]_0} \arrow[dr, "\text{“The constant”}"'] & & {[\underline{F - VSP \ with \ D - flag}]_0} \arrow[dl, "\text{Forget the D-flag}"] \\
 & {[\underline{VSP}_F]_0} &
\end{tikzcd}$$

Our D-flag is a Young diagram with two rows. Two boxes on top and one on the bottom. So we have a 3-dimensional vector space.

$$X_1 \rightarrowtail X_2 \rightarrowtail V$$

Dim X_1	Dim X_2		
0	0	$(0,0)$	1
0	1	$(0,0),(0,1)$	2
0	2	$(0,1),(0,2)$	2
0	3	$(0,2)$	1
1	1	$(0,0),(1,1)$	2
1	2	$(0,1),(1,1),(1,2)$	3
1	3	$(0,2),(1,2)$	2
2	2	$(1,1),(2,2)$	2
2	3	$(1,2),(2,2)$	2
3	3	$(2,2)$	1

So we have 6 pairs of three:

$$(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)$$

We are interested in "morphisms between representations of the quantum group". We have as objects "abelian categories of finite-dimensional representations of a given quantum group ($A_2 = SL(3)$). In particular, we can consider "hom-spaces in this abelian category". Picking a particular hom-space, we can groupoidify this vector space. So we pick two objects, i.e. Young diagrams.

Young diagrams represent functorial operations on representations. These are called the "Schur functors". These are operations built up from the symmetrized tensor power and the tensor product.

What I said:
We are going to be groupoidifying the hom-spaces between the values of the Schur functors at the tautological representation T.

What I should have said:
We are actually groupoidifying the hom-spaces between the Schur functors themselves.

"Schur functors"
"Schur-Weyl duality"
This is a duality between the representation theory of:

1. $GL(n)$ (often a Lie group)
2. $k!$ (a finite discrete group)

This relationship involves the study of $T_n \otimes \cdots \otimes T_n$ (k times) as a simultaneous representation of both of these groups, where T_n is the tautological representation of $GL(n)$. A Young diagram with k boxes is interpreted as a representation of $k!$. This turns into a notation for representations of $GL(n)$ by interpreting representations of $k!$ as abstract functorial operations...